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# 1-1 Enrichment

## *Significant Digits*

All measurements are approximations. The **significant digits** of an approximate number are those which indicate the results of a measurement. For example, the mass of an object, measured to the nearest gram, is 210 grams. The measurement 21$\underline{0}$ g has 3 significant digits. The mass of the same object, measured to the nearest 100 g, is 200 g. The measurement 200 g has one significant digit.

**1.** Nonzero digits and zeros between significant digits are significant. For example, the measurement 9.071 m has 4 significant digits, 9, 0, 7, and 1.

**2.** Zeros at the end of a decimal fraction are significant. The measurement 0.050 mm has 2 significant digits, 5 and 0.

**3.** Underlined zeros in whole numbers are significant. The measurement 104,0$\underline{0}$0 km has 5 significant digits, 1, 0, 4, 0, and 0.

In general, a computation involving multiplication or division of measurements *cannot* be more accurate than the least accurate measurement in the computation. Thus, the result of computation involving multiplication or division of measurements should be rounded to the number of significant digits in the least accurate measurement.

**Example** **The mass of 37 quarters is 21$\underline{0}$ g. Find the mass of one quarter.**

| | | |
|---|---|---|
| mass of 1 quarter | = 21$\underline{0}$ g ÷ 37 | 21$\underline{0}$ has 3 significant digits.<br>37 does not represent a measurement. |
| | = 5.68 g | Round the result to 3 significant digits.<br>Why? |

**Write the number of significant digits for each measurement.**

**1.** 8314.20 m **2.** 30.70 cm **3.** 0.01 mm **4.** 0.0605 mg

**5.** 37$\underline{0}$,000 km **6.** 370,0$\underline{0}$0 km **7.** $9.7 \times 10^4$ g **8.** $3.20 \times 10^{-2}$ g

**Solve. Round each result to the correct number of significant digits.**

**9.** 23 m × 1.54 m **10.** 12,0$\underline{0}$0 ft ÷ 520 ft **11.** 2.5 cm × 25

**12.** 11.01 mm × 11 **13.** 908 yd ÷ 0.5 **14.** 38.6 m × 4.0 m

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# 1-2 Enrichment

## *Properties of a Group*

A set of numbers forms a group with respect to an operation if for that operation the set has (1) the Closure Property, (2) the Associative Property, (3) a member which is an identity, and (4) an inverse for each member of the set.

**Example 1** **Does the set {0, 1, 2, 3, …} form a group with respect to addition?**

**Closure Property:** For all numbers in the set, is $a + b$ in the set? $0 + 1 = 1$, and 1 is in the set; $0 + 2 = 2$, and 2 is in the set; and so on. The set has closure for addition.

**Associative Property:** For all numbers in the set, does $a + (b + c) = (a + b) + c$? $0 + (1 + 2) = (0 + 1) + 2$; $1 + (2 + 3) = (1 + 2) + 3$; and so on. The set is associative for addition.

**Identity:** Is there some number, $i$, in the set such that $i + a = a = a + i$ for all $a$? $0 + 1 = 1 = 1 + 0$; $0 + 2 = 2 = 2 + 0$; and so on. The identity for addition is 0.

**Inverse:** Does each number, $a$, have an inverse, $a'$, such that $a' + a = a + a' = i$? The integer inverse of 3 is $-3$ since $-3 + 3 = 0$, and 0 is the identity for addition. But the set does not contain $-3$. Therefore, there is no inverse for 3.

The set is not a group with respect to addition because only three of the four properties hold.

**Example 2** **Is the set {−1, 1} a group with respect to multiplication?**

**Closure Property:** $(-1)(-1) = 1$; $(-1)(1) = -1$; $(1)(-1) = -1$; $(1)(1) = 1$
The set has closure for multiplication.

**Associative Property:** $(-1)[(-1)(-1)] = (-1)(1) = -1$; and so on
The set is associative for multiplication.

**Identity:** $1(-1) = -1$; $1(1) = 1$
The identity for multiplication is 1.

**Inverse:** $-1$ is the inverse of $-1$ since $(-1)(-1) = 1$, and 1 is the identity.
1 is the inverse of 1 since $(1)(1) = 1$, and 1 is the identity.
Each member has an inverse.

The set $\{-1, 1\}$ is a group with respect to multiplication because all four properties hold.

**Tell whether the set forms a group with respect to the given operation.**

**1.** {integers}, addition

**2.** {integers}, multiplication

**3.** $\left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \cdots\right\}$, addition

**4.** {multiples of 5}, multiplication

**5.** $\{x, x^2, x^3, x^4, \cdots\}$ addition

**6.** $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots\}$, multiplication

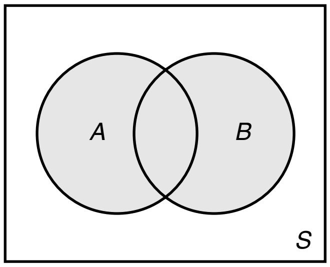
**7.** {irrational numbers}, addition

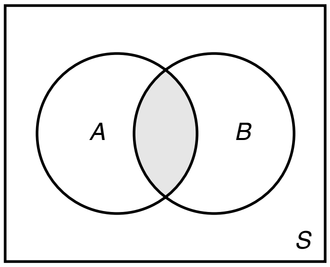
**8.** {rational numbers}, addition

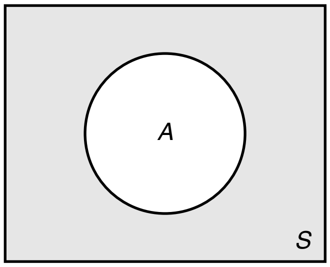
# 1-3 Enrichment

## *Venn Diagrams*

Relationships among sets can be shown using Venn diagrams. Study the diagrams below. The circles represent sets *A* and *B,* which are subsets of set *S*.



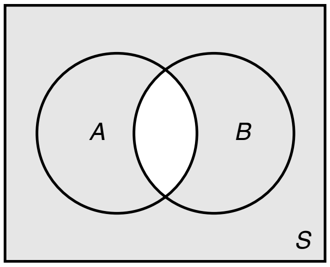




The union of *A* and *B* consists of all elements in *either A* or *B*.
The intersection of *A* and *B* consists of all elements in *both A* and *B*.
The complement of *A* consists of all elements *not* in *A*.

You can combine the operations of union, intersection, and finding the complement.

**Example** **Shade the region $(A \cap B)'$.**

$(A \cap B)'$ means the complement of the intersection of *A* and *B*. First find the intersection of *A* and *B*. Then find its complement.




**Draw a Venn diagram and shade the region indicated.**

**1.** $A' \cap B$

**2.** $A' \cup B$

**3.** $A' \cap B'$

**4.** $A' \cup B'$

**5.** $(A \cup B)'$

**6.** $A \cap B'$

**Draw a Venn diagram and three overlapping circles. Then shade the region indicated.**

**7.** $(A \cup B) \cup C'$

**8.** $(A \cup B)' \cap C'$

**9.** $A \cup (B \cup C)$

**10.** $(A \cup B) \cup C$

**11.** Is the union operation associative?

**12.** Is the intersection operation associative?

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# 1-4 Enrichment

## *Considering All Cases in Absolute Value Equations*

You have learned that absolute value equations with one set of absolute value symbols have two cases that must be considered. For example, $|x + 3| = 5$ must be broken into $x + 3 = 5$ or $-(x + 3) = 5$. For an equation with two sets of absolute value symbols, four cases must be considered.

Consider the problem $|x + 2| + 3 = |x + 6|$. First we must write the equations for the case where $x + 6 \geq 0$ and where $x + 6 < 0$. Here are the equations for these two cases:

$|x + 2| + 3 = x + 6$

$|x + 2| + 3 = -(x + 6)$

Each of these equations also has two cases. By writing the equations for both cases of each equation above, you end up with the following four equations:

$x + 2 + 3 = x + 6$ $\quad$ $x + 2 + 3 = -(x + 6)$

$-(x + 2) + 3 = x + 6$ $\quad$ $-x - 2 + 3 = -(x + 6)$

Solve each of these equations and check your solutions in the original equation, $|x + 2| + 3 = |x + 6|$. The only solution to this equation is $-\frac{5}{2}$.

**Solve each absolute value equation. Check your solution.**

**1.** $|x - 4| = |x + 7|$

**2.** $|2x + 9| = |x - 3|$

**3**. $|-3x - 6| = |5x + 10|$

**4.** $|x + 4| - 6 = |x - 3|$

**5.** How many cases would there be for an absolute value equation containing three sets of absolute value symbols?

**6.** List each case and solve $|x + 2| + |2x - 4| = |x - 3|$. Check your solution.

# 1-5 Enrichment

## *Equivalence Relations*

A relation R on a set $A$ is an *equivalence relation* if it has the following properties.

**Reflexive Property** For any element $a$ of set $A$, $a$ R $a$.

**Symmetric Property** For all elements $a$ and $b$ of set $A$, if $a$ R $b$, then $b$ R $a$.

**Transitive Property** For all elements $a$, $b$, and $c$ of set $A$, if $a$ R $b$ and $b$ R $c$, then $a$ R $c$.

Equality on the set of all real numbers is reflexive, symmetric, and transitive. Therefore, it is an equivalence relation.

**In each of the following, a relation and a set are given. Write *yes* if the relation is an equivalence relation on the given set. If it is not, tell which of the properties it fails to exhibit.**

**1.** $<$, {all numbers}

**2.** $\cong$, {all triangles in a plane}

**3.** is the sister of, {all women in Tennessee}

**4.** $\geq$, {all numbers}

**5.** is a factor of, {all nonzero integers}

**6.** $\sim$, {all polygons in a plane}

**7.** is the spouse of, {all people in Roanoke, Virginia}

**8.** $\perp$, {all lines in a plane}

**9.** is a multiple of, {all integers}

**10.** is the square of, {all numbers}

**11.** $\parallel$, {all lines in a plane}

**12.** has the same color eyes as, {all members of the Cleveland Symphony Orchestra}

**13.** is the greatest integer not greater than, {all numbers}

**14.** is the greatest integer not greater than, {all integers}

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# 1-6 Enrichment

## *Conjunctions and Disjunctions*

An absolute value inequality may be solved as a compound sentence.

**Example 1** **Solve $|2x| < 10$.**

$|2x| < 10$ means $2x < 10$ and $2x > -10$.

Solve each inequality. $x < 5$ and $x > -5$.

Every solution for $|2x| < 10$ is a replacement for $x$ that makes both $x < 5$ and $x > -5$ true.

A compound sentence that combines two statements by the word *and* is a *conjunction*.

**Example 2** **Solve $|3x - 7| \geq 11$.**

$|3x - 7| \geq 11$ means $3x - 7 \geq 11$ or $3x - 7 \leq -11$.

Solve each inequality. $3x \geq 18$ or $3x \leq -4$

$x \geq 6$ or $x \leq -\frac{4}{3}$

Every solution for the inequality is a replacement for $x$ that makes either $x \geq 6$ or $x \leq -\frac{4}{3}$ true.

A compound sentence that combines two statements by the word *or* is a *disjunction*.

**Solve each inequality. Then write whether the solution is a conjunction or disjunction.**

**1.** $|4x| > 24$

**2.** $|x - 7| \leq 8$

**3.** $|2x + 5| < 1$

**4.** $|x - 1| \geq 1$

**5.** $|3x - 1| \leq x$

**6.** $7 - |2x| > 5$

**7.** $\left|\frac{x}{2} + 1\right| \geq 7$

**8.** $\left|\frac{x - 4}{3}\right| < 4$

**9.** $|8 - x| > 2$

**10.** $|5 - 2x| \leq 3$

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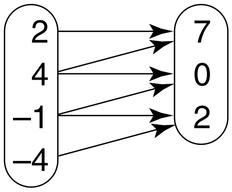
# 2-1 Enrichment

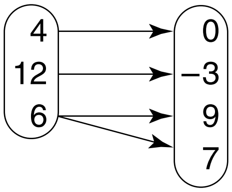
## *Mappings*

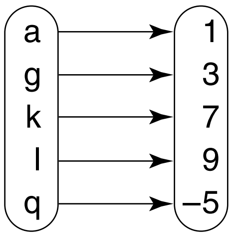
There are three special ways in which one set can be mapped to another. A set can be mapped *into* another set, *onto* another set, or can have a *one-to-one correspondence* with another set.

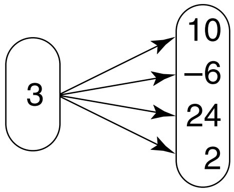
| | |
|---|---|
| **Into mapping** | A mapping from set *A* to set *B* where every element of *A* is mapped to one or more elements of set *B,* but never to an element not in *B.* |
| **Onto mapping** | A mapping from set *A* to set *B* where each element of set *B* has at least one element of set *A* mapped to it. |
| **One-to-one correspondence** | A mapping from set *A* onto set *B* where each element of set *A* is mapped to exactly one element of set *B* and different elements of *A* are never mapped to the same element of *B.* |

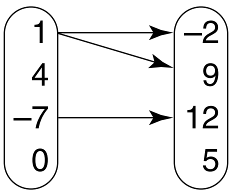
**State whether each set is mapped into the second set, onto the second set, or has a one-to-one correspondence with the second set.**

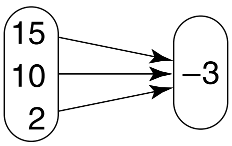
**1.** 

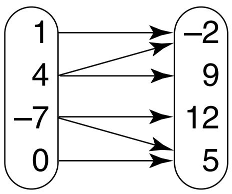

**2.** 

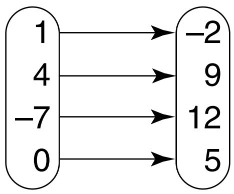

**3.** 


**4.** 


**5.** 


**6.** 


**7.** 


**8.** 


**9.** Can a set be mapped *onto* a set with fewer elements than it has?

**10.** Can a set be mapped *into* a set that has more elements than it has?

**11.** If a mapping from set *A* into set *B* is a one-to-one correspondence, what can you conclude about the number of elements in *A* and *B*?

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# 2-2 Enrichment

## *Greatest Common Factor*

Suppose we are given a linear equation $ax + by = c$ where $a$, $b$, and $c$ are nonzero integers, and we want to know if there exist *integers* $x$ and $y$ that satisfy the equation. We could try guessing a few times, but this process would be time consuming for an equation such as $588x + 432y = 72$. By using the Euclidean Algorithm, we can determine not only if such integers $x$ and $y$ exist, but also find them. The following example shows how this algorithm works.

**Example** **Find integers $x$ and $y$ that satisfy $588x + 432y = 72$.**

Divide the greater of the two coefficients by the lesser to get a quotient and remainder. Then, repeat the process by dividing the divisor by the remainder until you get a remainder of 0. The process can be written as follows.

$588 = 432(1) + 156 \qquad (1)$
$432 = 156(2) + 120 \qquad (2)$
$156 = 120(1) + 36 \qquad (3)$
$120 = 36(3) + 12 \qquad (4)$
$36 = 12(3)$

The last nonzero remainder is the GCF of the two coefficients. If the constant term 72 is divisible by the GCF, then integers $x$ and $y$ do exist that satisfy the equation. To find $x$ and $y$, work backward in the following manner.

$72 = 6 \cdot 12$
$= 6 \cdot [120 - 36(3)]$ Substitute for 12 using (4)
$= 6(120) - 18(36)$
$= 6(120) - 18[156 - 120(1)]$ Substitute for 36 using (3)
$= -18(156) + 24(120)$
$= -18(156) + 24[432 - 156(2)]$ Substitute for 120 using (2)
$= 24(432) - 66(156)$
$= 24(432) - 66[588 - 432(1)]$ Substitute for 156 using (1)
$= 588(-66) + 432(90)$

Thus, $x = -66$ and $y = 90$.

**Find integers $x$ and $y$, if they exist, that satisfy each equation.**

**1.** $27x + 65y = 3$

**2.** $45x + 144y = 36$

**3.** $90x + 117y = 10$

**4.** $123x + 36y = 15$

**5.** $1032x + 1001y = 1$

**6.** $3125x + 3087y = 1$

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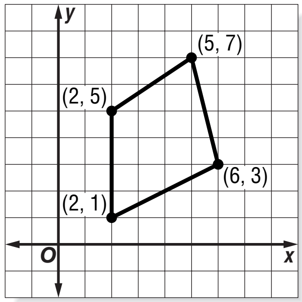
# 2-3 Enrichment

## *Aerial Surveyors and Area*

Many land regions have irregular shapes. Aerial surveyors supply aerial mappers with lists of coordinates and elevations for the areas that need to be photographed from the air. These maps provide information about the horizontal and vertical features of the land.




**Step 1** List the ordered pairs for the vertices in counterclockwise order, repeating the first ordered pair at the bottom of the list.

**Step 2** Find $D$, the sum of the downward diagonal products (from left to right).

$D = (5 \cdot 5) + (2 \cdot 1) + (2 \cdot 3) + (6 \cdot 7)$
$\quad = 25 + 2 + 6 + 42$ or $75$

(5, 7)
(2, 5)
(2, 1)
(6, 3)
(5, 7)

**Step 3** Find $U$, the sum of the upward diagonal products (from left to right).

$U = (2 \cdot 7) + (2 \cdot 5) + (6 \cdot 1) + (5 \cdot 3)$
$\quad = 14 + 10 + 6 + 15$ or $45$

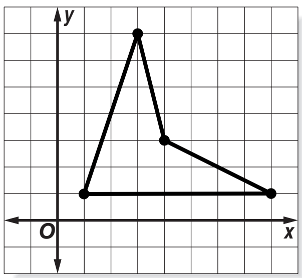
**Step 4** Use the formula $A = \frac{1}{2}(D - U)$ to find the area.

$A = \frac{1}{2}(75 - 45)$
$\quad = \frac{1}{2}(30)$ or $15$

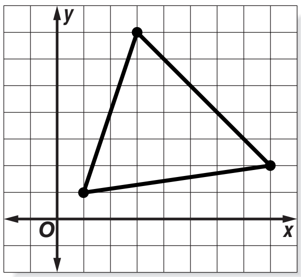
The area is 15 square units. Count the number of square units enclosed by the polygon. Does this result seem reasonable?

**Use the coordinate method to find the area of each region in square units.**

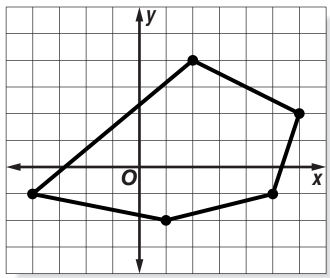
**1.**




**2.**




**3.**



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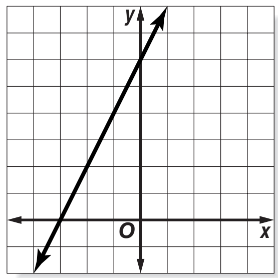
# 2-4 Enrichment

## *Two-Intercept Form of a Linear Equation*

You are already familiar with the slope-intercept form of a linear equation, $y = mx + b$. Linear equations can also be written in the form $\frac{x}{a} + \frac{y}{b} = 1$ with $x$-intercept $a$ and $y$-intercept $b$. This is called two-intercept form.

**Example 1** **Draw the graph of $\frac{x}{-3} + \frac{y}{6} = 1$.**

The graph crosses the $x$-axis at $-3$ and the $y$-axis at 6. Graph $(-3, 0)$ and $(0, 6)$, then draw a straight line through them.




**Example 2** **Write $3x + 4y = 12$ in two-intercept form.**

$\frac{3x}{12} + \frac{4y}{12} = \frac{12}{12}$ Divide by 12 to obtain 1 on the right side.

$\frac{x}{4} + \frac{y}{3} = 1$ Simplify.

The $x$-intercept is 4; the $y$-intercept is 3.

**Use the given intercepts a and b, to write an equation in two-intercept form. Then draw the graph.**

**1.** $a = -2, b = -4$

**2.** $a = 1, b = 8$

**3.** $a = 3, b = 5$

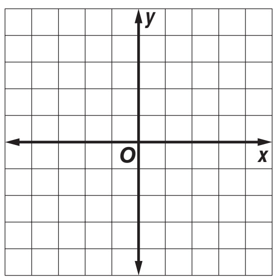
**4.** $a = 6, b = 9$

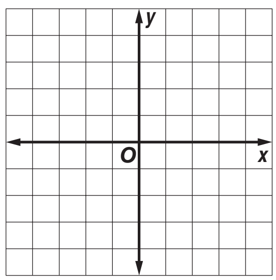
**Write each equation in two-intercept form. Then draw the graph.**

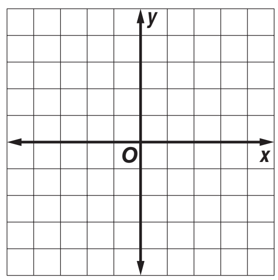
**5.** $3x - 2y = -6$

**6.** $\frac{1}{2}x + \frac{1}{4}y = 1$

**7.** $5x + 2y = -10$



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# 2-5 Enrichment

## *Median-Fit Lines*

A **median-fit line** is a particular type of line of fit. Follow the steps below to find the equation of the median-fit line for the data.

| Approximate Percentage of Violent Crimes Committed by Juveniles That Victims Reported to Law Enforcement | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Year** | 1980 | 1982 | 1984 | 1986 | 1988 | 1990 | 1992 | 1994 | 1996 |
| **Offenders** | 36 | 35 | 33 | 32 | 31 | 30 | 29 | 29 | 30 |

**Source:** U.S. Bureau of Justice Statistics

**1.** Divide the data into three approximately equal groups. There should always be the same number of points in the first and third groups. In this case, there will be three data points in each group.

**2.** Find $x_1$, $x_2$, and $x_3$, the medians of the $x$ values in groups 1, 2, and 3, respectively. Find $y_1$, $y_2$, and $y_3$, the medians of the $y$ values in groups 1, 2, and 3, respectively.

**3.** Find an equation of the line through $(x_1, y_1)$ and $(x_3, y_3)$.

**4.** Find $Y$, the $y$-coordinate of the point on the line in Exercise 2 with an $x$-coordinate of $x_2$.

**5.** The median-fit line is parallel to the line in Exercise 2, but is one-third closer to $(x_2, y_2)$. This means it passes through $\left(x_2, \frac{2}{3}Y + \frac{1}{3}y_2\right)$. Find this ordered pair.

**6.** Write an equation of the median-fit line.

**7.** Use the median-fit line to predict the percentage of juvenile violent crime offenders in 2010 and 2020.

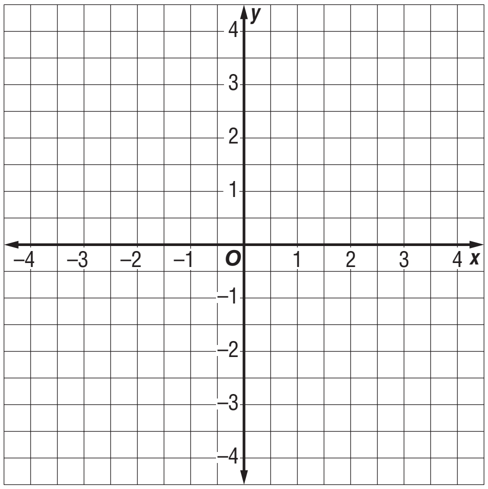
# 2-6 Enrichment

## *Greatest Integer Functions*

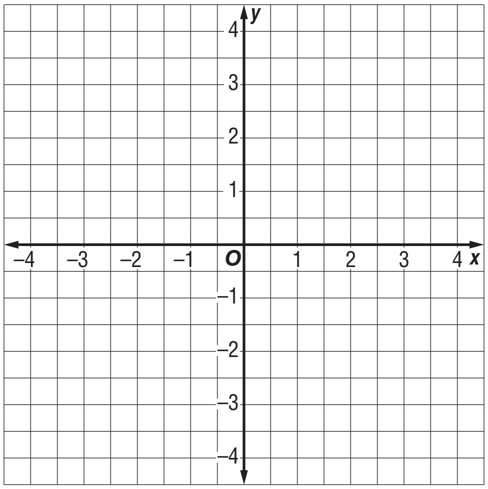
Use the greatest integer function $\llbracket x \rrbracket$ to explore some unusual graphs. It will be helpful to make a chart of values for each functions and to use a colored pen or pencil.

**Graph each function.**

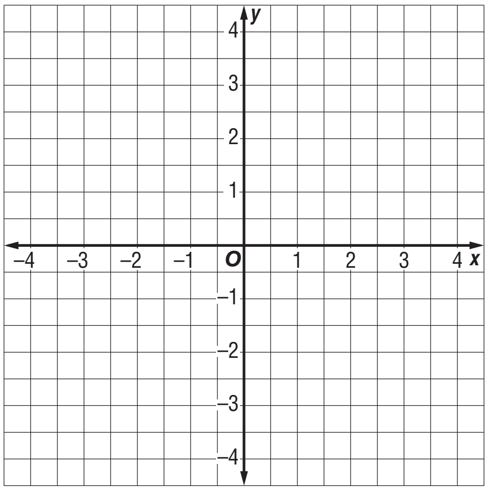
**1.** $y = 2x - \llbracket x \rrbracket$



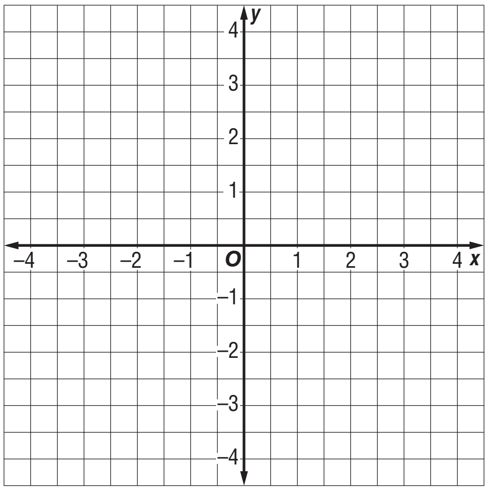

**2.** $y = \dfrac{\llbracket x \rrbracket}{\llbracket x \rrbracket}$




**3.** $y = \dfrac{\llbracket 0.5x + 1 \rrbracket}{\llbracket 0.5x + 1 \rrbracket}$




**4.** $y = \dfrac{x}{\llbracket x \rrbracket}$



# 2-7 Enrichment

## *Algebraic Proof*

The following paragraph states a result you might be asked to prove in a mathematics course. Parts of the paragraph are numbered.

01 Let $n$ be a positive integer.

02 Also, let $n_1 = s(n_1)$ be the sum of the squares of the digits in $n$.

03 Then $n_2 = s(n_1)$ is the sum of the squares of the digits of $n_1$, and $n_3 = s(n_2)$ is the sum of the squares of the digits of $n_2$.

04 In general, $n_k = s(n_{k-1})$ is the sum of the squares of the digits of $n_{k-1}$.

05 Consider the sequence: $n, n_1, n_2, n_3, \cdots, n_k, \cdots$.

06 In this sequence either all the terms from some $k$ on have the value 1,

07 or some term, say $n_j$, has the value 4, so that the eight terms 4, 16, 37, 58, 89, 145, 42, and 20 keep repeating from that point on.

**Use the paragraph to answer these questions.**

**1.** Use the sentence in line 01. List the first five values of $n$.

**2.** Use 9246 for $n$ and give an example to show the meaning of line 02.

**3.** In line 02, which symbol shows a function? Explain the function in a sentence.

**4.** For $n = 9246$, find $n_2$ and $n_3$ as described in sentence 03.

**5.** How do the first four sentences relate to sentence 05?

**6.** Use $n = 31$ and find the first four terms of the sequence.

**7.** Which sentence of the paragraph is illustrated by $n = 31$?

**8.** Use $n = 61$ and find the first ten terms.

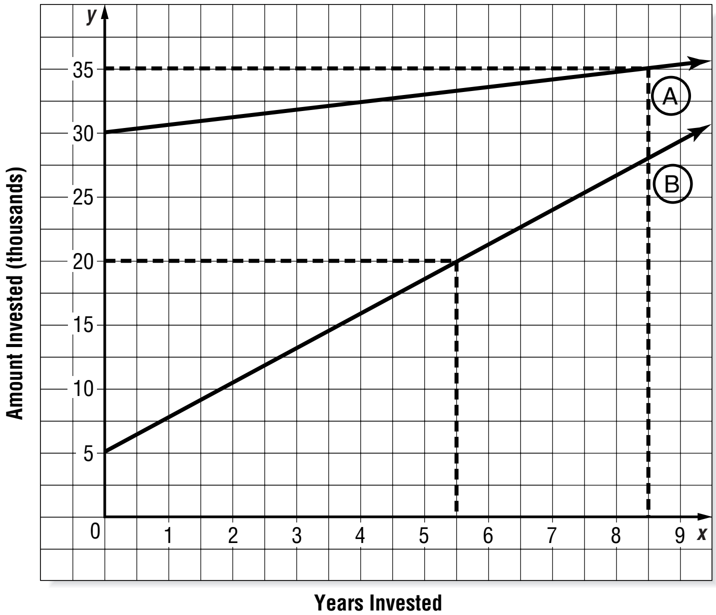
**9.** Which sentence is illustrated by $n = 61$?

# 3-1 Enrichment

## *Investments*

The following graph shows the value of two different investments over time. Line A represents an initial investment of $30,000 with a bank paying passbook savings interest. Line B represents an initial investment of $5,000 in a profitable mutual fund with dividends reinvested and capital gains accepted in shares. By deriving the linear equation $y = mx + b$ for A and B, you can predict the value of these investments for years to come.




**1.** The $y$-intercept, $b$, is the initial investment. Find $b$ for each of the following.

**a.** line A

**b.** line B

**2.** The slope of the line, $m$, is the rate of return. Find $m$ for each of the following.

**a.** line A

**b.** line B

**3.** What are the equations of each of the following lines?

**a.** line A

**b.** line B

**4.** What will be the value of the mutual fund after 11 years of investment?

**5.** What will be the value of the bank account after 11 years of investment?

**6.** When will the mutual fund and the bank account have equal value?

**7.** Which investment has the greatest payoff after 11 years of investment?

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# 3-2 Enrichment

## *Using Coordinates*

From one observation point, the line of sight to a downed plane is given by $y = x - 1$. This equation describes the distance from the observation point to the plane in a straight line. From another observation point, the line of sight is given by $x + 3y = 21$. What are the coordinates of the point at which the crash occurred?

Solve the system of equations $\begin{cases} y = x - 1 \\ x + 3y = 21 \end{cases}$.

$$\begin{aligned} x + 3y &= 21 \\ x + 3(x - 1) &= 21 && \text{Substitute } x - 1 \text{ for } y. \\ x + 3x - 3 &= 21 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

$$\begin{aligned} x + 3y &= 21 \\ 6 + 3y &= 21 && \text{Substitute 6 for } x. \\ 3y &= 15 \\ y &= 5 \end{aligned}$$

The coordinates of the crash are (6, 5).

**Solve the following.**

**1.** The lines of sight to a forest fire are as follows.

From Ranger Station A: $3x + y = 9$
From Ranger Station B: $2x + 3y = 13$
Find the coordinates of the fire.

**2.** An airplane is traveling along the line $x - y = -1$ when it sees another airplane traveling along the line $5x + 3y = 19$. If they continue along the same lines, at what point will their flight paths cross?

**3.** Two mine shafts are dug along the paths of the following equations.

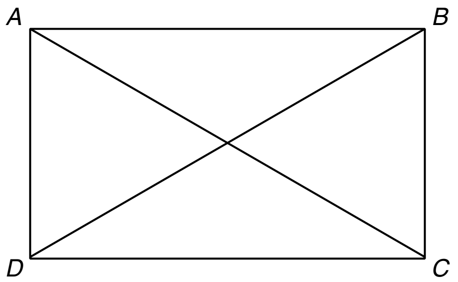
$x - y = 1400$
$2x + y = 1300$

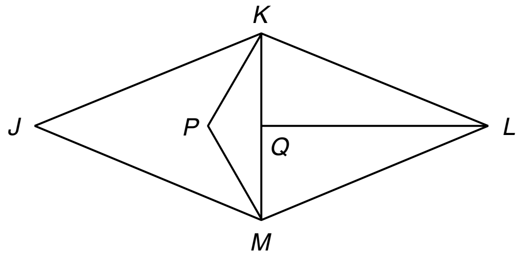
If the shafts meet at a depth of 200 feet, what are the coordinates of the point at which they meet?

# 3-3 Enrichment

## *Tracing Strategy*

Try to trace over each of the figures below without tracing the same segment twice.

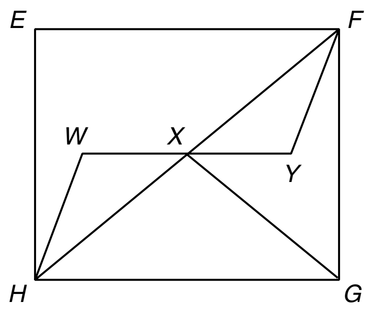


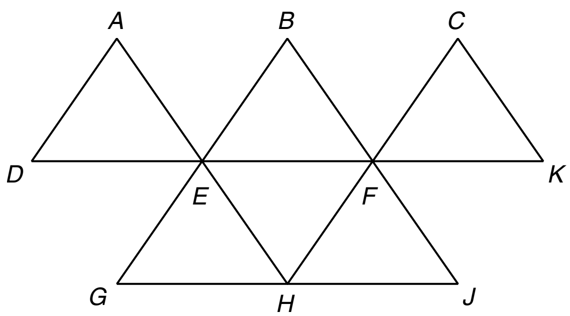

The figure at the left cannot be traced, but the one at the right can. The rule is that a figure is traceable if it has no more than two points where an odd number of segments meet. The figure at the left has three segments meeting at each of the four corners. However, the figure at the right has only two points, $L$ and $Q$, where an odd number of segments meet.

**Determine if each figure can be traced without tracing the same segment twice. If it can, then name the starting point and name the segments in the order they should be traced.**

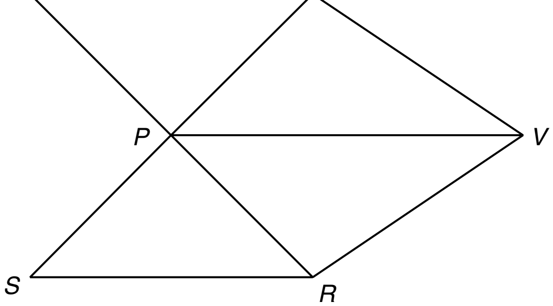
**1.**




**2.**




**3.**



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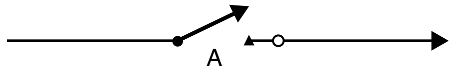
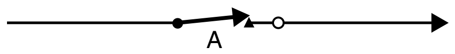
# 3-4 Enrichment

## *Computer Circuits and Logic*

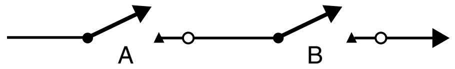
Computers operate according to the laws of logic. The circuits of a computer can be described using logic.

1. 

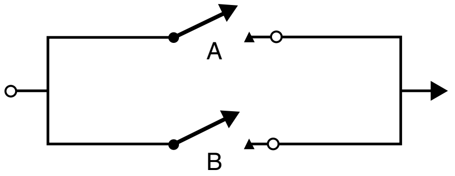
With switch A open, no current flows. The value 0 is assigned to an open switch.

2. 

With switch A closed, current flows. The value 1 is assigned to a closed switch.

3. 

With switches A and B open, no current flows. This circuit can be described by the conjunction, A · B.

4. 

In this circuit, current flows if either A or B is closed. This circuit can be described by the disjunction, A + B.

| A | B | A + B |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Truth tables are used to describe the flow of current in a circuit. The table at the left describes the circuit in diagram 4. According to the table, the only time current does not flow through the circuit is when both switches A and B are open.

**Draw a circuit diagram for each of the following.**

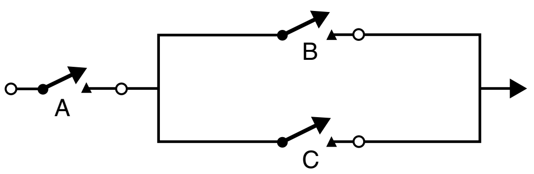
**1.** (A · B) + C

**2.** (A + B) · C

**3.** (A + B) · (C + D)

**4.** (A · B) + (C · D)

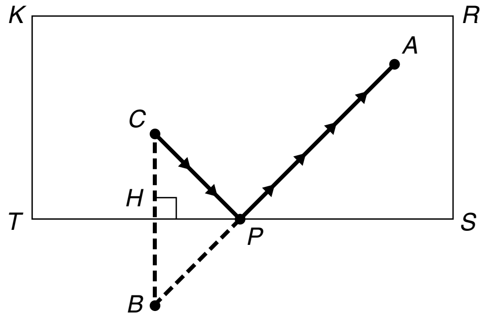
**5.** Construct a truth table for the following circuit.



# 3-5 Enrichment

## *Billiards*

The figure at the right shows a billiard table. The object is to use a cue stick to strike the ball at point $C$ so that the ball will hit the sides (or cushions) of the table at least once before hitting the ball located at point $A$. In playing the game, you need to locate point $P$.



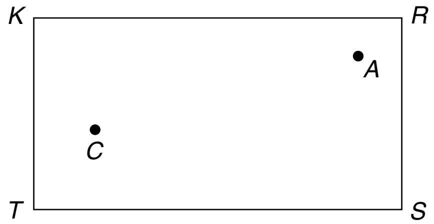

**Step 1** Find point $B$ so that $\overline{BC} \perp \overline{ST}$ and $\overline{BH} \cong \overline{CH}$. $B$ is called the reflected image of $C$ in $\overline{ST}$.

**Step 2** Draw $\overline{AB}$.

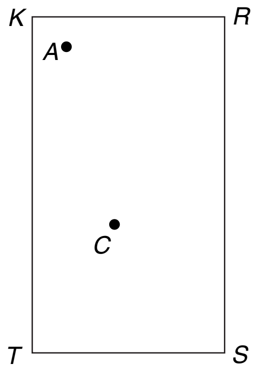
**Step 3** $\overline{AB}$ intersects $\overline{ST}$ at the desired point $P$.

**For each billiards problem, the cue ball at point $C$ must strike the indicated cushion(s) and then strike the ball at point $A$. Draw and label the correct path for the cue ball using the process described above.**

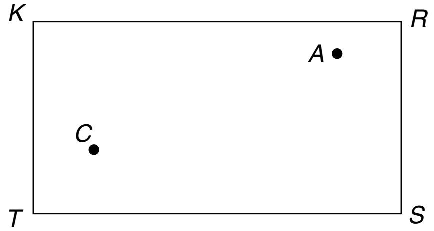
**1.** cushion $\overline{KR}$



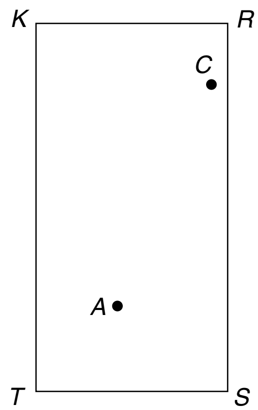

**2.** cushion $\overline{RS}$




**3.** cushion $\overline{TS}$, then cushion $\overline{RS}$




**4.** cushion $\overline{KT}$, then cushion $\overline{RS}$



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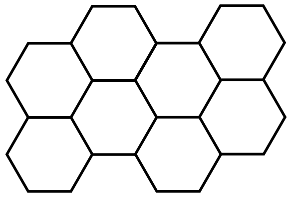
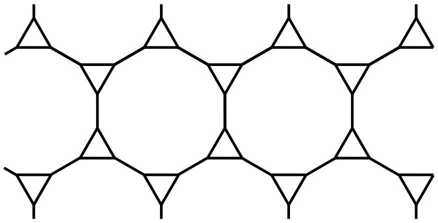
# 4-1 Enrichment

## *Tessellations*

A **tessellation** is an arangement of polygons covering a plane without any gaps or overlapping. One example of a tessellation is a honeycomb. Three congruent regular hexagons meet at each vertex, and there is no wasted space between cells. This tessellation is called a regular tessellation since it is formed by congruent regular polygons.



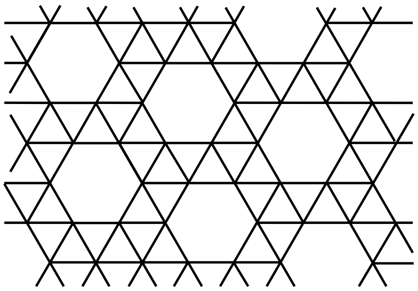
A **semi-regular tessellation** is a tessellation formed by two or more regular polygons such that the number of sides of the polygons meeting at each vertex is the same.



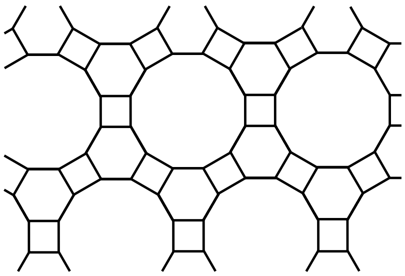
For example, the tessellation at the left has two regular dodecagons and one equilateral triangle meeting at each vertex. We can name this tessellation a 3-12-12 for the number of sides of each polygon that meet at one vertex.

**Name each semi-regular tessellation shown according to the number of sides of the polygons that meet at each vertex.**

**1.**

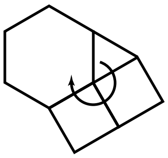


**2.**

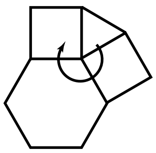


**An equilateral triangle, two squares, and a regular hexagon can be used to surround a point in two different orders. Continue each pattern to see which is a semi-regular tessellation.**

**3.** 3-4-4-6



**4.** 3-4-6-4



**On another sheet of paper, draw part of each design. Then determine if it is a semi-regular tessellation.**

**5.** 3-3-4-12 **6.** 3-4-3-12 **7.** 4-8-8 **8.** 3-3-3-4-4

# 4-2 Enrichment

## *Sundaram's Sieve*

The properties and patterns of prime numbers have fascinated many mathematicians. In 1934, a young East Indian student named Sundaram constructed the following matrix.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4 | 7 | 10 | 13 | 16 | 19 | 22 | 25 | . . . |
| 7 | 12 | 17 | 22 | 27 | 32 | 37 | 42 | . . . |
| 10 | 17 | 24 | 31 | 38 | 45 | 52 | 59 | . . . |
| 13 | 22 | 31 | 40 | 49 | 58 | 67 | 76 | . . . |
| 16 | 27 | 38 | 49 | 60 | 71 | 82 | 93 | . . . |
| . | . | . | . | . | . | . | . | . . . |

A surprising property of this matrix is that it can be used to determine whether or not some numbers are prime.

**Complete these problems to discover this property.**

**1.** The first row and the first column are created by using an arithmetic pattern. What is the common difference used in the pattern?

**2.** Find the next four numbers in the first row.

**3.** What are the common differences used to create the patterns in rows 2, 3, 4, and 5?

**4.** Write the next two rows of the matrix. Include eight numbers in each row.

**5.** Choose any five numbers from the matrix. For each number $n$, that you chose from the matrix, find $2n + 1$.

**6.** Write the factorization of each value of $2n + 1$ that you found in problem 5.

**7.** Use your results from problems 5 and 6 to complete this statement: If $n$ occurs in the matrix, then $2n + 1$ ____________ (is/is not) a prime number.

**8.** Choose any five numbers that are not in the matrix. Find $2n + 1$ for each of these numbers. Show that each result is a prime number.

**9.** Complete this statement: If $n$ does not occur in the matrix, then $2n + 1$ is ______________________.

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# 4-3 Enrichment

## *Fourth-Order Determinants*

To find the value of a 4 × 4 determinant, use a method called **expansion by minors**.

First write the expansion. Use the first row of the determinant.
Remember that the signs of the terms alternate.

$$\begin{vmatrix} 6 & -3 & 2 & 7 \\ 0 & 4 & 3 & 5 \\ 0 & 2 & 1 & -4 \\ 6 & 0 & -2 & 0 \end{vmatrix} = 6\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & -4 \\ 0 & -2 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 3 & 5 \\ 0 & 1 & -4 \\ 6 & -2 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & -4 \\ 6 & 0 & 0 \end{vmatrix} - 7\begin{vmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 6 & 0 & -2 \end{vmatrix}$$

Then evaluate each 3 × 3 determinant. Use any row.

$$\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & -4 \\ 0 & -2 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix}$$
$$= 2(-16 - 10)$$
$$= -52$$

$$\begin{vmatrix} 0 & 3 & 5 \\ 0 & 1 & -4 \\ 6 & -2 & 0 \end{vmatrix} = -3\begin{vmatrix} 0 & -4 \\ 6 & 0 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 6 & -2 \end{vmatrix}$$
$$= -3(24) + 5(-6)$$
$$= -102$$

$$\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & -4 \\ 6 & 0 & 0 \end{vmatrix} = 6\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix}$$
$$= 6(-16 - 10)$$
$$= -156$$

$$\begin{vmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 6 & 0 & -2 \end{vmatrix} = -4\begin{vmatrix} 0 & 1 \\ 6 & -2 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix}$$
$$= -4(-6) + 3(-12)$$
$$= -12$$

Finally, evaluate the original 4 × 4 determinant.

$$\begin{vmatrix} 6 & -3 & 2 & 7 \\ 0 & 4 & 3 & 5 \\ 0 & 2 & 1 & -4 \\ 6 & 0 & -2 & 0 \end{vmatrix} = 6(-52) + 3(-102) + 2(-156) - 7(-12) = -846$$

**Evaluate each determinant.**

**1.** $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 4 & 3 & -1 & 0 \\ 2 & -5 & 4 & 4 \\ 1 & -2 & 0 & 2 \end{vmatrix}$

**2.** $\begin{vmatrix} 3 & 3 & 3 & 3 \\ 2 & 1 & 2 & 1 \\ 4 & 3 & -1 & 5 \\ 2 & 5 & 0 & 1 \end{vmatrix}$

**3.** $\begin{vmatrix} 1 & 4 & 3 & 0 \\ -2 & -3 & 6 & 4 \\ 5 & 1 & 1 & 2 \\ 4 & 2 & 5 & -1 \end{vmatrix}$

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# 4-4 Enrichment

## *Properties of Determinants*

The following properties often help when evaluating determinants.

- If all the elements of a row (or column) are zero, the value of the determinant is zero.

  $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$

- Multiplying all the elements of a row (or column) by a constant is equivalent to multiplying the value of the determinant by the constant.

  $3\begin{vmatrix} 4 & -1 \\ 5 & 3 \end{vmatrix} = \begin{vmatrix} 12 & -3 \\ 5 & 3 \end{vmatrix}$

- If two rows (or columns) have equal corresponding elements, the value of the determinant is zero.

  $\begin{vmatrix} 5 & 5 \\ -3 & -3 \end{vmatrix} = 0$

- The value of a determinant is unchanged if any multiple of a row (or column) is added to corresponding elements of another row (or column).

  $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix}$

  (Row 2 is added to row 1.)

- If two rows (or columns) are interchanged, the sign of the determinant is changed.

  $\begin{vmatrix} 4 & 5 \\ -3 & 8 \end{vmatrix} = \begin{vmatrix} -3 & 8 \\ 4 & 5 \end{vmatrix}$

- The value of the determinant is unchanged if row 1 is interchanged with column 1, and row 2 is interchanged with column 2. The result is called the transpose.

  $\begin{vmatrix} 5 & -7 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ -7 & 4 \end{vmatrix}$

**Exercises 1–6**

Verify each property above by evaluating the given determinants and give another example of the property.

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# 4-5 Enrichment

## *Matrix Multiplication*

A furniture manufacturer makes upholstered chairs and wood tables. Matrix $A$ shows the number of hours spent on each item by three different workers. One day the factory receives an order for 10 chairs and 3 tables. This is shown in matrix $B$.

$$\begin{array}{l} \\ \\ \text{chair} \\ \text{table} \end{array} \begin{array}{c} \begin{array}{ccc} & \text{hours} & \\ \text{woodworker} & \text{finsher} & \text{upholsterer} \end{array} \\ \begin{bmatrix} 4 & 2 & 12 \\ 18 & 15 & 0 \end{bmatrix} \end{array} = A \qquad \begin{array}{r} \\ \text{number ordered} \end{array} \begin{array}{c} \begin{array}{cc} \text{chair} & \text{table} \end{array} \\ \begin{bmatrix} 10 & 3 \end{bmatrix} \end{array} = B$$

$$\begin{bmatrix} 10 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 12 \\ 18 & 15 & 0 \end{bmatrix} = [10(4) + 3(18) \quad 10(2) + 3(15) \quad 10(12) + 3(0)] = [94 \quad 65 \quad 120]$$

The product of the two matrices shows the number of hours needed for each type of worker to complete the order: 94 hours for woodworking, 65 hours for finishing, and 120 hours for upholstering.

To find the total labor cost, multiply by a matrix that shows the hourly rate for each worker: \$15 for woodworking, \$9 for finishing, and \$12 for upholstering.

$$C = \begin{bmatrix} 15 \\ 9 \\ 12 \end{bmatrix} [94 \quad 65 \quad 120] = [94(15) + 65(9) + 120(12)] = \$3435$$

**Use matrix multiplication to solve these problems.**

A candy company packages caramels, chocolates, and hard candy in three different assortments: traditional, deluxe, and superb. For each type of candy the table below gives the number in each assortment, the number of Calories per piece, and the cost to make each piece.

| | traditional | deluxe | superb | Calories per piece | cost per piece (cents) |
|---|---|---|---|---|---|
| caramels | 10 | 16 | 15 | 60 | 10 |
| chocolates | 12 | 8 | 25 | 70 | 12 |
| hard candy | 10 | 16 | 8 | 55 | 6 |

The company receives an order for 300 traditional, 180 deluxe and 100 superb assortments.

**1.** Find the number of each type of candy needed to fill the order.

**2.** Find the total number of Calories in each type of assortment.

**3.** Find the cost of production for each type of assortment.

**4.** Find the cost to fill the order.

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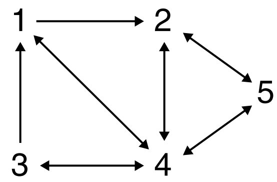
# 4-6 Enrichment

## *Communications Networks*




The diagram at the right represents a communications network linking five computer remote stations. The arrows indicate the direction in which signals can be transmitted and received by each computer. We can generate a matrix to describe this network.

$$A = \begin{array}{r} \\ \text{from} \\ \text{computer } i \end{array} \overset{\text{to computer } j}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}}$$

The entry in position $a_{ij}$ represents the number of ways to send a message from computer $i$ to computer $j$ directly. Compare the entries of matrix $A$ to the diagram to verify the entries. For example, there is one way to send a message from computer 3 to computer 4, so $A_{3,4} = 1$. A computer cannot send a message to itself, so $A_{1,1} = 0$.

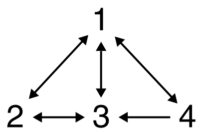
Matrix $A$ is a communications network for direct communication. Suppose you want to send a message from one computer to another using exactly one other computer as a relay point. It can be shown that the entries of matrix $A^2$ represent the number of ways to send a message from one point to another by going through a third station. For example, a message may be sent from station 1 to station 5 by going through station 2 or station 4 on the way. Therefore, ${A^2}_{1,5} = 2$.

$$A^2 = \begin{array}{r} \\ \text{from} \\ \text{computer } i \end{array} \overset{\text{to computer } j}{\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 4 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}}$$

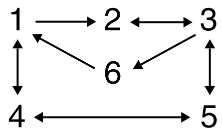
Again, compare the entries of matrix $A^2$ to the communications diagram to verify that the entries are correct. Matrix $A^2$ represents using exactly one relay.

**For each network, find the matrices $A$ and $A^2$. Then write the number of ways the messages can be sent for each matrix.**

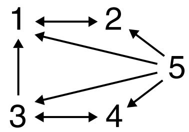
**1.**




**2.**




**3.**



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# 4-7 Enrichment

## *Permutation Matrices*

A permutation matrix is a square matrix in which each row and each column has one entry that is 1. All the other entries are 0. Find the inverse of a permutation matrix interchanging the rows and columns. For example, row 1 is interchanged with column 1, row 2 is interchanged with column 2.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$P$ is a $4 \times 4$ permutation matrix. $P^{-1}$ is the inverse of $P$.

**Solve each problem.**

**1.** There is just one $2 \times 2$ permutation matrix that is not also an identity matrix. Write this matrix.

**2.** Find the inverse of the matrix you wrote in Exercise 1. What do you notice?

**3.** Show that the two matrices in Exercises 1 and 2 are inverses.

**4.** Write the inverse of this matrix.

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

**5.** Use $B^{-1}$ from problem 4. Verify that $B$ and $B^{-1}$ are inverses.

**6.** Permutation matrices can be used to write and decipher codes. To see how this is done, use the message matrix $M$ and matrix $B$ from problem 4. Find matrix $C$ so that $C$ equals the product $MB$. Use the rules below.

0 times a letter = 0
1 times a letter = the same letter
0 plus a letter = the same letter

$$M = \begin{bmatrix} \text{S} & \text{H} & \text{E} \\ \text{S} & \text{A} & \text{W} \\ \text{H} & \text{I} & \text{M} \end{bmatrix}$$

**7.** Now find the product $CB^{-1}$. What do you notice?

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# 4-8 Enrichment

## *Properties of Matrices*

Computing with matrices is different from computing with real numbers. Stated below are some properties of the real number system. Are these also true for matrices? In the problems on this page, you will investigate this question.

For all real numbers $a$ and $b$, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Multiplication is commutative. For all real numbers $a$ and $b$, $ab = ba$.

Multiplication is associative. For all real numbers $a$, $b$, and $c$, $a(bc) = (ab)c$.

**Use the matrices *A*, *B*, and *C* for the problems. Write whether each statement is true. Assume that a 2-by-2 matrix is the 0 matrix if and only if all of its elements are zero.**

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

**1.** $AB = 0$

**2.** $AC = 0$

**3.** $BC = 0$

**4.** $AB = BA$

**5.** $AC = CA$

**6.** $BC = CB$

**7.** $A(BC) = (AB)C$

**8.** $B(CA) = (BC)A$

**9.** $B(AC) = (BA)C$

**10.** Write a statement summarizing your findings about the properties of matrix multiplication.

 

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# 5-1 Enrichment

## *Properties of Exponents*

The rules about powers and exponents are usually given with letters such as *m, n,* and *k* to represent exponents. For example, one rule states that $a^m \cdot a^n = a^{m+n}$.

In practice, such exponents are handled as algebraic expressions and the rules of algebra apply.

**Example 1** **Simplify $2a^2(a^{n+1} + a^{4n})$.**

$$2a^2(a^{n+1} + a^{4n}) = 2a^2 \cdot a^{n+1} + 2a^2 \cdot a^{4n}$$ Use the Distributive Law.

$$= 2a^{2+n+1} + 2a^{2+4n}$$ Recall $a^m \cdot a^n = a^{m+n}$.

$$= 2a^{n+3} + 2a^{2+4n}$$ Simplify the exponent $2 + n + 1$ as $n + 3$.

It is important always to collect *like* terms only.

**Example 2** **Simplify $(a^n + b^m)^2$.**

$$(a^n + b^m)^2 = (a^n + b^m)(a^n + b^m)$$

$$= \underbrace{a^n \cdot a^n}_{F} + \underbrace{a^n \cdot b^m}_{O} + \underbrace{a^n \cdot b^m}_{I} + \underbrace{b^m \cdot b^m}_{L}$$ The second and third terms are like terms.

$$= a^{2n} + 2a^nb^m + b^{2m}$$

**Simplify each expression by performing the indicated operations.**

**1.** $2^3 2^m$

**2.** $(a^3)^n$

**3.** $(4^n b^2)^k$

**4.** $(x^3 a^j)^m$

**5.** $(-ay^n)^3$

**6.** $(-b^k x)^2$

**7.** $(c^2)^{hk}$

**8.** $(-2d^n)^5$

**9.** $(a^2b)(a^nb^2)$

**10.** $(x^ny^m)(x^my^n)$

**11.** $\frac{a^n}{2}$

**12.** $\frac{12x^3}{4x^n}$

**13.** $(ab^2 - a^2b)(3a^n + 4b^n)$

**14.** $ab^2(2a^2b^{n-1} + 4ab^n + 6b^{n+1})$

# 5-2 Enrichment

## *Polynomials with Fractional Coefficients*

Polynomials may have fractional coefficients as long as there are no variables in the denominators. Computing with fractional coefficients is performed in the same way as computing with whole-number coefficients.

**Simpliply. Write all coefficients as fractions.**

**1.** $\left(\frac{3}{5}m - \frac{2}{7}p - \frac{1}{3}n\right) - \left(\frac{7}{3}p - \frac{5}{2}m - \frac{3}{4}n\right)$

**2.** $\left(\frac{3}{2}x - \frac{4}{3}y - \frac{5}{4}z\right) + \left(-\frac{1}{4}x + y + \frac{2}{5}z\right) + \left(-\frac{7}{8}x - \frac{6}{7}y + \frac{1}{2}z\right)$

**3.** $\left(\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2\right) + \left(\frac{5}{6}a^2 + \frac{2}{3}ab - \frac{3}{4}b^2\right)$

**4.** $\left(\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2\right) - \left(\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{5}{6}b^2\right)$

**5.** $\left(\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2\right) \cdot \left(\frac{1}{2}a - \frac{2}{3}b\right)$

**6.** $\left(\frac{2}{3}a^2 - \frac{1}{5}a + \frac{2}{7}\right) \cdot \left(\frac{2}{3}a^3 + \frac{1}{5}a^2 - \frac{2}{7}a\right)$

**7.** $\left(\frac{2}{3}x^2 - \frac{3}{4}x - 2\right) \cdot \left(\frac{4}{5}x - \frac{1}{6}x^2 - \frac{1}{2}\right)$

**8.** $\left(\frac{1}{6} + \frac{1}{3}x + \frac{1}{6}x^4 - \frac{1}{2}x^2\right) \cdot \left(\frac{1}{6}x^3 - \frac{1}{3} - \frac{1}{3}x\right)$

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# 5-3 Enrichment

## *Oblique Asymptotes*

The graph of $y = ax + b$, where $a \neq 0$, is called an oblique asymptote of $y = f(x)$ if the graph of $f$ comes closer and closer to the line as $x \to \infty$ or $x \to -\infty$. $\infty$ is the mathematical symbol for **infinity**, which means *endless*.

For $f(x) = 3x + 4 + \frac{2}{x}$, $y = 3x + 4$ is an oblique asymptote because $f(x) - 3x - 4 = \frac{2}{x}$, and $\frac{2}{x} \to 0$ as $x \to \infty$ or $-\infty$. In other words, as $|x|$ increases, the value of $\frac{2}{x}$ gets smaller and smaller approaching 0.

**Example** **Find the oblique asymptote for $f(x) = \frac{x^2 + 8x + 15}{x + 2}$.**

$$\begin{array}{r|rrr} -2 & 1 & 8 & 15 \\ & & -2 & -12 \\ \hline & 1 & 6 & 3 \end{array}$$

Use synthetic division.

$$y = \frac{x^2 + 8x + 15}{x + 2} = x + 6 + \frac{3}{x + 2}$$

As $|x|$ increases, the value of $\frac{3}{x + 2}$ gets smaller. In other words, since $\frac{3}{x + 2} \to 0$ as $x \to \infty$ or $x \to -\infty$, $y = x + 6$ is an oblique asymptote.

**Use synthetic division to find the oblique asymptote for each function.**

**1.** $y = \frac{8x^2 - 4x + 11}{x + 5}$

**2.** $y = \frac{x^2 + 3x - 15}{x - 2}$

**3.** $y = \frac{x^2 - 2x - 18}{x - 3}$

**4.** $y = \frac{ax^2 + bx + c}{x - d}$

**5.** $y = \frac{ax^2 + bx + c}{x + d}$

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# 5-4 Enrichment

## *Using Patterns to Factor*

Study the patterns below for factoring the sum and the difference of cubes.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

This pattern can be extended to other odd powers. Study these examples.

**Example 1** **Factor $a^5 + b^5$.**

Extend the first pattern to obtain $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

**Check:** $(a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$

$$\begin{array}{r} a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 \phantom{+ b^5} \\ \underline{\phantom{a^5} + a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5} \\ = a^5 \qquad\qquad\qquad\qquad\qquad + b^5 \end{array}$$

**Example 2** **Factor $a^5 - b^5$.**

Extend the second pattern to obtain $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

**Check:** $(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4$

$$\begin{array}{r} a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 \phantom{- b^5} \\ \underline{\phantom{a^5} - a^4b - a^3b^2 - a^2b^3 - ab^4 - b^5} \\ = a^5 \qquad\qquad\qquad\qquad\qquad - b^5 \end{array}$$

In general, if $n$ is an odd integer, when you factor $a^n + b^n$ or $a^n - b^n$, one factor will be either $(a + b)$ or $(a - b)$, depending on the sign of the original expression. The other factor will have the following properties:

- The first term will be $a^{n-1}$ and the last term will be $b^{n-1}$.
- The exponents of $a$ will decrease by 1 as you go from left to right.
- The exponents of $b$ will increase by 1 as you go from left to right.
- The degree of each term will be $n - 1$.
- If the original expression was $a^n + b^n$, the terms will alternately have + and − signs.
- If the original expression was $a^n - b^n$, the terms will all have + signs.

**Use the patterns above to factor each expression.**

**1.** $a^7 + b^7$

**2.** $c^9 - d^9$

**3.** $e^{11} + f^{11}$

**To factor $x^{10} - y^{10}$, change it to $(x^5 + y^5)(x^5 - y^5)$ and factor each binomial. Use this approach to factor each expression.**

**4.** $x^{10} - y^{10}$

**5.** $a^{14} - b^{14}$

# 5-5 Enrichment

## *Approximating Square Roots*

Consider the following expansion.

$$\left(a + \frac{b}{2a}\right)^2 = a^2 + \frac{2ab}{2a} + \frac{b^2}{4a^2}$$

$$= a^2 + b + \frac{b^2}{4a^2}$$

Think what happens if $a$ is very great in comparison to $b$. The term $\frac{b^2}{4a^2}$ is very small and can be disregarded in an approximation.

$$\left(a + \frac{b}{2a}\right)^2 \approx a^2 + b$$

$$a + \frac{b}{2a} \approx \sqrt{a^2 + b}$$

Suppose a number can be expressed as $a^2 + b$, $a > b$. Then an approximate value of the square root is $a + \frac{b}{2a}$. You should also see that $a - \frac{b}{2a} \approx \sqrt{a^2 - b}$.

**Example** **Use the formula $\sqrt{a^2 \pm b} \approx a \pm \frac{b}{2a}$ to approximate $\sqrt{101}$ and $\sqrt{622}$.**

**a.** $\sqrt{101} = \sqrt{100 + 1} = \sqrt{10^2 + 1}$

Let $a = 10$ and $b = 1$.

$\sqrt{101} \approx 10 + \frac{1}{2(10)}$

$\approx 10.05$

**b.** $\sqrt{622} = \sqrt{625 - 3} = \sqrt{25^2 - 3}$

Let $a = 25$ and $b = 3$.

$\sqrt{622} \approx 25 - \frac{3}{2(25)}$

$\approx 24.94$

**Use the formula to find an approximation for each square root to the nearest hundredth. Check your work with a calculator.**

**1.** $\sqrt{626}$

**2.** $\sqrt{99}$

**3.** $\sqrt{402}$

**4.** $\sqrt{1604}$

**5.** $\sqrt{223}$

**6.** $\sqrt{80}$

**7.** $\sqrt{4890}$

**8.** $\sqrt{2505}$

**9.** $\sqrt{3575}$

**10.** $\sqrt{1,441,100}$

**11.** $\sqrt{290}$

**12.** $\sqrt{260}$

**13.** Show that $a - \frac{b}{2a} \approx \sqrt{a^2 - b}$ for $a > b$.

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# 5-6 Enrichment

## *Special Products with Radicals*

Notice that $(\sqrt{3})(\sqrt{3}) = 3$, or $(\sqrt{3})^2 = 3$.

In general, $(\sqrt{x})^2 = x$ when $x \geq 0$.

Also, notice that $(\sqrt{9})(\sqrt{4}) = \sqrt{36}$.

In general, $(\sqrt{x})(\sqrt{y}) = \sqrt{xy}$ when $x$ and $y$ are not negative.

You can use these ideas to find the special products below.

$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

$(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + 2\sqrt{ab} + (\sqrt{b})^2 = a + 2\sqrt{ab} + b$

$(\sqrt{a} - \sqrt{b})^2 = (\sqrt{a})^2 - 2\sqrt{ab} + (\sqrt{b})^2 = a - 2\sqrt{ab} + b$

*Example 1* **Find the product: $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$.**

$(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5}) = (\sqrt{2})^2 - (\sqrt{5})^2 = 2 - 5 = -3$

*Example 2* **Evaluate $(\sqrt{2} + \sqrt{8})^2$.**

$$(\sqrt{2} + \sqrt{8})^2 = (\sqrt{2})^2 + 2\sqrt{2}\sqrt{8} + (\sqrt{8})^2$$
$$= 2 + 2\sqrt{16} + 8 = 2 + 2(4) + 8 = 2 + 8 + 8 = 18$$

**Multiply.**

**1.** $(\sqrt{3} - \sqrt{7})(\sqrt{3} + \sqrt{7})$

**2.** $(\sqrt{10} + \sqrt{2})(\sqrt{10} - \sqrt{2})$

**3.** $(\sqrt{2x} - \sqrt{6})(\sqrt{2x} - \sqrt{6})$

**4.** $\left(\sqrt{3} - (-7)\right)^2$

**5.** $(\sqrt{1000} + \sqrt{10})^2$

**6.** $(\sqrt{y} + \sqrt{5})(\sqrt{y} - \sqrt{5})$

**7.** $(\sqrt{50} - \sqrt{x})^2$

**8.** $(\sqrt{x} + 20)^2$

You can extend these ideas to patterns for sums and differences of cubes.
Study the pattern below.

$(\sqrt[3]{8} - \sqrt[3]{x})(\sqrt[3]{8^2} + \sqrt[3]{8x} + \sqrt[3]{x^2}) = \sqrt[3]{8^3} - \sqrt[3]{x^3} = 8 - x$

**Multiply.**

**9.** $(\sqrt[3]{2} - \sqrt[3]{5})(\sqrt[3]{2^2} + \sqrt[3]{10} + \sqrt[3]{5^2})$

**10.** $(\sqrt[3]{y} + \sqrt[3]{w})(\sqrt[3]{y^2} - \sqrt[3]{yw} + \sqrt[3]{w^2})$

**11.** $(\sqrt[3]{7} + \sqrt[3]{20})(\sqrt[3]{7^2} - \sqrt[3]{140} + \sqrt[3]{20^2})$

**12.** $(\sqrt[3]{11} - \sqrt[3]{8})(\sqrt[3]{11^2} + \sqrt[3]{88} + \sqrt[3]{8^2})$

# 5-7 Enrichment

## *Lesser-Known Geometric Formulas*

Many geometric formulas involve radical expressions.

**Make a drawing to illustrate each of the formulas given on this page. Then evaluate the formula for the given value of the variable. Round answers to the nearest hundredth.**

**1.** The area of an isosceles triangle. Two sides have length $a$; the other side has length $c$. Find $A$ when $a = 6$ and $c = 7$.

$A = \frac{c}{4}\sqrt{4a^2 - c^2}$

**2.** The area of an equilateral triangle with a side of length $a$. Find $A$ when $a = 8$.

$A = \frac{a^2}{4}\sqrt{3}$

**3.** The area of a regular pentagon with a side of length $a$. Find $A$ when $a = 4$.

$A = \frac{a^2}{4}\sqrt{25 + 10\sqrt{5}}$

**4.** The area of a regular hexagon with a side of length $a$. Find $A$ when $a = 9$.

$A = \frac{3a^2}{2}\sqrt{3}$

**5.** The volume of a regular tetrahedron with an edge of length $a$. Find $V$ when $a = 2$.

$V = \frac{a^3}{12}\sqrt{2}$

**6.** The area of the curved surface of a right cone with an altitude of $h$ and radius of base $r$. Find $S$ when $r = 3$ and $h = 6$.

$S = \pi r\sqrt{r^2 + h^2}$

**7.** Heron's Formula for the area of a triangle uses the semi-perimeter $s$, where $s = \frac{a + b + c}{2}$. The sides of the triangle have lengths $a$, $b$, and $c$. Find $A$ when $a = 3$, $b = 4$, and $c = 5$.

$A = \sqrt{s(s - a)(s - b)(s - c)}$

**8.** The radius of a circle inscribed in a given triangle also uses the semi-perimeter. Find $r$ when $a = 6$, $b = 7$, and $c = 9$.

$r = \frac{\sqrt{s(s - a)(s - b)(s - c)}}{s}$

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# 5-8 Enrichment

## *Truth Tables*

In mathematics, the basic operations are addition, subtraction, multiplication, division, finding a root, and raising to a power. In logic, the basic operations are the following: *not* ($\sim$), *and* ($\wedge$), *or* ($\vee$), and *implies* ($\rightarrow$).

If $P$ and $Q$ are statements, then $\sim P$ means not $P$; $\sim Q$ means not $Q$; $P \wedge Q$ means $P$ and $Q$; $P \vee Q$ means $P$ or $Q$; and $P \rightarrow Q$ means $P$ implies $Q$. The operations are defined by truth tables. On the left below is the truth table for the statement $\sim P$. Notice that there are two possible conditions for $P$, true ($T$) or false ($F$). If $P$ is true, $\sim P$ is false; if $P$ is false, $\sim P$ is true. Also shown are the truth tables for $P \wedge Q$, $P \vee Q$, and $P \rightarrow Q$.

| $P$ | $\sim P$ |
|---|---|
| $T$ | $F$ |
| $F$ | $T$ |

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

| $P$ | $Q$ | $P \rightarrow Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

You can use this information to find out under what conditions a complex statement is true.

*Example* **Under what conditions is $\sim P \vee Q$ true?**

Create the truth table for the statement. Use the information from the truth table above for $P \vee Q$ to complete the last column.

| $P$ | $Q$ | $\sim P$ | $\sim P \vee Q$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ |

The truth table indicates that $\sim P \vee Q$ is true in all cases except where $P$ is true and $Q$ is false.

**Use truth tables to determine the conditions under which each statement is true.**

**1.** $\sim P \vee \sim Q$

**2.** $\sim P \rightarrow (P \rightarrow Q)$

**3.** $(P \vee Q) \vee (\sim P \wedge \sim Q)$

**4.** $(P \rightarrow Q) \vee (Q \rightarrow P)$

**5.** $(P \rightarrow Q) \wedge (Q \rightarrow P)$

**6.** $(\sim P \wedge \sim Q) \rightarrow \sim(P \vee Q)$

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# 5-9 Enrichment

## *Conjugates and Absolute Value*

When studying complex numbers, it is often convenient to represent a complex number by a single variable. For example, we might let $z = x + y\boldsymbol{i}$. We denote the conjugate of $z$ by $\bar{z}$. Thus, $\bar{z} = x - y\boldsymbol{i}$.

We can define the absolute value of a complex number as follows.

$$|z| = |x + y\boldsymbol{i}| = \sqrt{x^2 + y^2}$$

There are many important relationships involving conjugates and absolute values of complex numbers.

**Example 1** **Show $|z|^2 = z\bar{z}$ for any complex number $z$.**

Let $z = x + y\boldsymbol{i}$. Then,

$$\begin{aligned} z &= (x + y\boldsymbol{i})(x - y\boldsymbol{i}) \\ &= x^2 + y^2 \\ &= \sqrt{(x^2 + y^2)^2} \\ &= |z|^2 \end{aligned}$$

**Example 2** **Show $\dfrac{\bar{z}}{|z|^2}$ is the multiplicative inverse for any nonzero complex number $z$.**

We know $|z|^2 = z\bar{z}$. If $z \neq 0$, then we have $z\left(\dfrac{\bar{z}}{|z|^2}\right) = 1$.

Thus, $\dfrac{\bar{z}}{|z|^2}$ is the multiplicative inverse of $z$.

**For each of the following complex numbers, find the absolute value and multiplicative inverse.**

**1.** $2\boldsymbol{i}$

**2.** $-4 - 3\boldsymbol{i}$

**3.** $12 - 5\boldsymbol{i}$

**4.** $5 - 12\boldsymbol{i}$

**5.** $1 + \boldsymbol{i}$

**6.** $\sqrt{3} - \boldsymbol{i}$

**7.** $\dfrac{\sqrt{3}}{3} + \dfrac{\sqrt{3}}{3}\boldsymbol{i}$

**8.** $\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}\boldsymbol{i}$

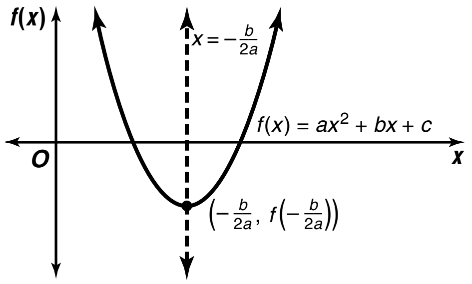
**9.** $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\boldsymbol{i}$

# 6-1 Enrichment

## *Finding the Axis of Symmetry of a Parabola*

As you know, if $f(x) = ax^2 + bx + c$ is a quadratic function, the values of $x$ that make $f(x)$ equal to zero are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The average of these two number values is $-\frac{b}{2a}$. The function $f(x)$ has its maximum or minimum value when $x = -\frac{b}{2a}$. Since the axis of symmetry of the graph of $f(x)$ passes through the point where the maximum or minimum occurs, the axis of symmetry has the equation $x = -\frac{b}{2a}$.




**Example** **Find the vertex and axis of symmetry for $f(x) = 5x^2 + 10x - 7$.**

Use $x = -\frac{b}{2a}$.

$x = -\frac{10}{2(5)} = -1$ The $x$-coordinate of the vertex is $-1$.

Substitute $x = -1$ in $f(x) = 5x^2 + 10x - 7$.

$f(-1) = 5(-1)^2 + 10(-1) - 7 = -12$

The vertex is $(-1, -12)$.

The axis of symmetry is $x = -\frac{b}{2a}$, or $x = -1$.

**Find the vertex and axis of symmetry for the graph of each function using $x = -\frac{b}{2a}$.**

**1.** $f(x) = x^2 - 4x - 8$

**2.** $g(x) = -4x^2 - 8x + 3$

**3.** $y = -x^2 + 8x + 3$

**4.** $f(x) = 2x^2 + 6x + 5$

**5.** $A(x) = x^2 + 12x + 36$

**6.** $k(x) = -2x^2 + 2x - 6$

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# 6-2 Enrichment

## *Graphing Absolute Value Equations*

You can solve absolute value equations in much the same way you solved quadratic equations. Graph the related absolute value function for each equation using a graphing calculator. Then use the ZERO feature in the CALC menu to find its real solutions, if any. Recall that solutions are points where the graph intersects the $x$-axis.

**For each equation, make a sketch of the related graph and find the solutions rounded to the nearest hundredth.**

**1.** $|x + 5| = 0$

**2.** $|4x - 3| + 5 = 0$

**3.** $|x - 7| = 0$

**4.** $|x + 3| - 8 = 0$

**5.** $-|x + 3| + 6 = 0$

**6.** $|x - 2| - 3 = 0$

**7.** $|3x + 4| = 2$

**8.** $|x + 12| = 10$

**9.** $|x| - 3 = 0$

**10.** Explain how solving absolute value equations algebraically and finding zeros of absolute value functions graphically are related.

# 6-3 Enrichment

## *Euler's Formula for Prime Numbers*

Many mathematicians have searched for a formula that would generate prime numbers. One such formula was proposed by Euler and uses a quadratic polynomial, $x^2 + x + 41$.

**Find the values of $x^2 + x + 41$ for the given values of $x$. State whether each value of the polynomial is or is not a prime number.**

**1.** $x = 0$

**2.** $x = 1$

**3.** $x = 2$

**4.** $x = 3$

**5.** $x = 4$

**6.** $x = 5$

**7.** $x = 6$

**8.** $x = 17$

**9.** $x = 28$

**10.** $x = 29$

**11.** $x = 30$

**12.** $x = 35$

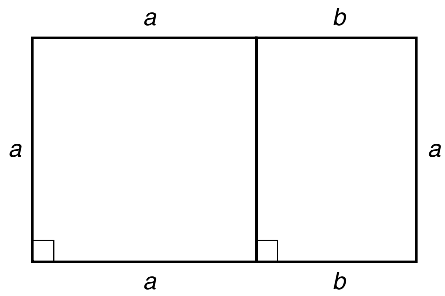
**13.** Does the formula produce all prime numbers greater than 40? Give examples in your answer.

**14.** Euler's formula produces primes for many values of $x$, but it does not work for all of them. Find the first value of $x$ for which the formula fails. (*Hint:* Try multiples of ten.)

# 6-4 Enrichment

## *The Golden Quadratic Equations*

A **golden rectangle** has the property that its length can be written as $a + b$, where $a$ is the width of the rectangle and $\frac{a+b}{a} = \frac{a}{b}$. Any golden rectangle can be divided into a square and a smaller golden rectangle, as shown.




The proportion used to define golden rectangles can be used to derive two quadratic equations. These are sometimes called *golden quadratic equations*.

**Solve each problem.**

1. In the proportion for the golden rectangle, let $a$ equal 1. Write the resulting quadratic equation and solve for $b$.

2. In the proportion, let $b$ equal 1. Write the resulting quadratic equation and solve for $a$.

3. Describe the difference between the two golden quadratic equations you found in exercises 1 and 2.

4. Show that the positive solutions of the two equations in exercises 1 and 2 are reciprocals.

5. Use the Pythagorean Theorem to find a radical expression for the diagonal of a golden rectangle when $a = 1$.

6. Find a radical expression for the diagonal of a golden rectangle when $b = 1$.

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# 6-5 Enrichment

## *Sum and Product of Roots*

Sometimes you may know the roots of a quadratic equation without knowing the equation itself. Using your knowledge of factoring to solve an equation, you can work backward to find the quadratic equation. The rule for finding the sum and product of roots is as follows:

| | |
|---|---|
| **Sum and Product of Roots** | If the roots of $ax^2 + bx + c = 0$, with $a \neq 0$, are $s_1$ and $s_2$, then $s_1 + s_2 = -\frac{b}{a}$ and $s_1 \cdot s_2 = \frac{c}{a}$. |

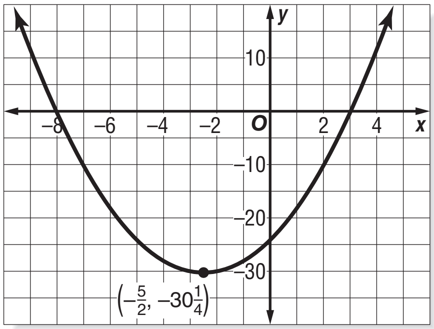
*Example* **A road with an initial gradient, or slope, of 3% can be represented by the formula $y = ax^2 + 0.03x + c$, where $y$ is the elevation and $x$ is the distance along the curve. Suppose the elevation of the road is 1105 feet at points 200 feet and 1000 feet along the curve. You can find the equation of the transition curve. Equations of transition curves are used by civil engineers to design smooth and safe roads.**

The roots are $x = 3$ and $x = -8$.

$3 + (-8) = -5$ Add the roots.

$3(-8) = -24$ Multiply the roots.

Equation: $x^2 + 5x - 24 = 0$




**Write a quadratic equation that has the given roots.**

**1.** 6, $-9$ **2.** 5, $-1$ **3.** 6, 6

**4.** $4 \pm \sqrt{3}$ **6.** $-\frac{2}{5}, \frac{2}{7}$ **6.** $\frac{-2 \pm 3\sqrt{5}}{7}$

**Find *k* such that the number given is a root of the equation.**

**7.** 7; $2x^2 + kx - 21 = 0$ **8.** $-2$; $x^2 - 13x + k = 0$

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# 6-6 Enrichment

## *Patterns with Differences and Sums of Squares*

Some whole numbers can be written as the difference of two squares and some cannot. Formulas can be developed to describe the sets of numbers algebraically.

**If possible, write each number as the difference of two squares. Look for patterns.**

| | | | |
|---|---|---|---|
| **1.** 0 | **2.** 1 | **3.** 2 | **4.** 3 |
| **5.** 4 | **6.** 5 | **7.** 6 | **8.** 7 |
| **9.** 8 | **10.** 9 | **11.** 10 | **12.** 11 |
| **13.** 12 | **14.** 13 | **15.** 14 | **16.** 15 |

**Even numbers can be written as $2n$, where $n$ is one of the numbers 0, 1, 2, 3, and so on. Odd numbers can be written $2n + 1$. Use these expressions for these problems.**

**17.** Show that any odd number can be written as the difference of two squares.

**18.** Show that the even numbers can be divided into two sets: those that can be written in the form $4n$ and those that can be written in the form $2 + 4n$.

**19.** Describe the even numbers that cannot be written as the difference of two squares.

**20.** Show that the other even numbers can be written as the difference of two squares.

**Every whole number can be written as the sum of squares. It is never necessary to use more than four squares. Show that this is true for the whole numbers from 0 through 15 by writing each one as the sum of the least number of squares.**

| | | |
|---|---|---|
| **21.** 0 | **22.** 1 | **23.** 2 |
| **24.** 3 | **25.** 4 | **26.** 5 |
| **27.** 6 | **28.** 7 | **29.** 8 |
| **30.** 9 | **31.** 10 | **32.** 11 |
| **33.** 12 | **34.** 13 | **35.** 14 |
| **36.** 15 | | |

# 6-7 Enrichment

## *Graphing Absolute Value Inequalities*

You can solve absolute value inequalities by graphing in much the same manner you graphed quadratic inequalities. Graph the related absolute function for each inequality by using a graphing calculator. For $>$ and $\geq$, identify the $x$-values, if any, for which the graph lies *below* the $x$-axis. For $<$ and $\leq$, identify the $x$ values, if any, for which the graph lies *above* the $x$-axis.

**For each inequality, make a sketch of the related graph and find the solutions rounded to the nearest hundredth.**

**1.** $|x - 3| > 0$

**2.** $|x| - 6 < 0$

**3.** $|x + 4| - 8 > 0$

**4.** $2|x + 6| - 2 \geq 0$

**5.** $|3x - 3| \geq 0$

**6.** $|x - 7| < 5$

**7.** $|7x - 1| > 13$

**8.** $|x - 3.6| \leq 4.2$

**9.** $|2x + 5| \leq 7$

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# 7-1 Enrichment

## *Approximation by Means of Polynomials*

Many scientific experiments produce pairs of numbers $[x, f(x)]$ that can be related by a formula. If the pairs form a function, you can fit a polynomial to the pairs in exactly one way. Consider the pairs given by the following table.

| $x$ | 1 | 2 | 4 | 7 |
|---|---|---|---|---|
| $f(x)$ | 6 | 11 | 39 | −54 |

We will assume the polynomial is of degree three. Substitute the given values into this expression.

$$f(x) = A + B(x - x_0) + C(x - x_0)(x - x_1) + D(x - x_0)(x - x_1)(x - x_2)$$

You will get the system of equations shown below. You can solve this system and use the values for $A$, $B$, $C$, and $D$ to find the desired polynomial.

$$6 = A$$
$$11 = A + B(2 - 1) = A + B$$
$$39 = A + B(4 - 1) + C(4 - 1)(4 - 2) = A + 3B + 6C$$
$$-54 = A + B(7 - 1) + C(7 - 1)(7 - 2) + D(7 - 1)(7 - 2)(7 - 4) = A + 6B + 30C + 90D$$

**Solve.**

**1.** Solve the system of equations for the values $A$, $B$, $C$, and $D$.

**2.** Find the polynomial that represents the four ordered pairs. Write your answer in the form $y = a + bx + cx^2 + dx^3$.

**3.** Find the polynomial that gives the following values.

| $x$ | 8 | 12 | 15 | 20 |
|---|---|---|---|---|
| $f(x)$ | −207 | 169 | 976 | 3801 |

**4.** A scientist measured the volume $f(x)$ of carbon dioxide gas that can be absorbed by one cubic centimeter of charcoal at pressure $x$. Find the values for $A$, $B$, $C$, and $D$.

| $x$ | 120 | 340 | 534 | 698 |
|---|---|---|---|---|
| $f(x)$ | 3.1 | 5.5 | 7.1 | 8.3 |

# 7-2 Enrichment

## *Golden Rectangles*

**Use a straightedge, a compass, and the instructions below to construct a golden rectangle.**

**1.** Construct square *ABCD* with sides of 2 centimeters.

**2.** Construct the midpoint of $\overline{AB}$. Call the midpoint *M*.

**3.** Using *M* as the center, set your compass opening at *MC*. Construct an arc with center *M* that intersects $\overline{AB}$. Call the point of intersection *P*.

**4.** Construct a line through *P* that is perpendicular to $\overline{AB}$.

**5.** Extend $\overline{DC}$ so that it intersects the perpendicular. Call the intersection point *Q*. *APQD* is a golden rectangle. Check this conclusion by finding the value of $\frac{AP}{QP}$ .

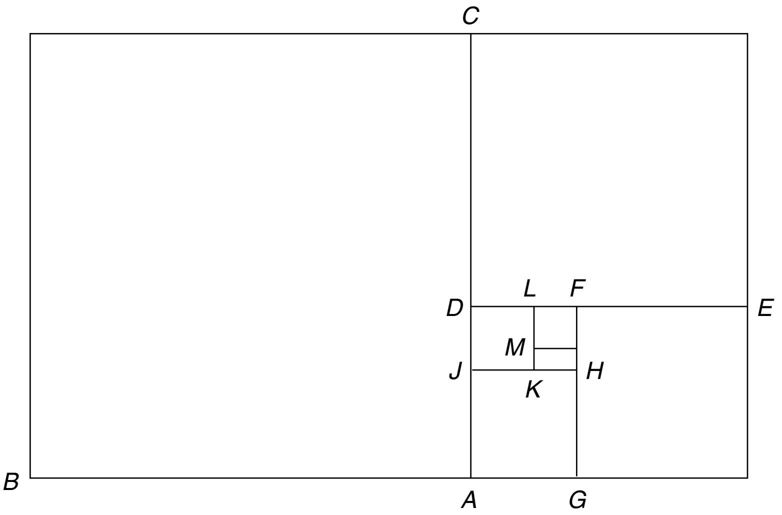
**A figure consisting of similar golden rectangles is shown below. Use a compass and the instructions below to draw quarter-circle arcs that form a spiral like that found in the shell of a chambered nautilus.**

**6.** Using *A* as a center, draw an arc that passes through *B* and *C*.

**7.** Using *D* as a center, draw an arc that passes through *C* and *E*.

**8.** Using *F* as a center, draw an arc that passes through *E* and *G*.

**9.** Continue drawing arcs, using *H*, *K*, and *M* as the centers.



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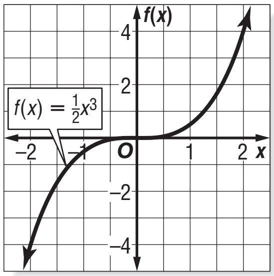
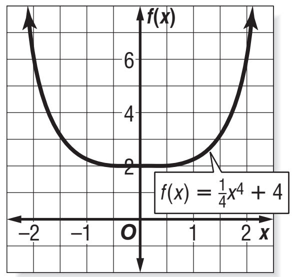
# 7-3 Enrichment

## *Odd and Even Polynomial Functions*

Functions whose graphs are symmetric with respect to the origin are called *odd* functions. If $f(-x) = -f(x)$ for all $x$ in the domain of $f(x)$, then $f(x)$ is odd.




Functions whose graphs are symmetric with respect to the $y$-axis are called *even* functions. If $f(-x) = f(x)$ for all $x$ in the domain of $f(x)$, then $f(x)$ is even.

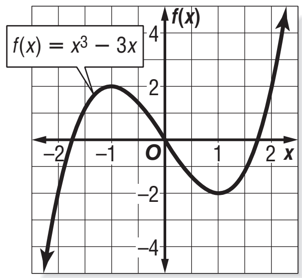



**Example** **Determine whether $f(x) = x^3 - 3x$ is odd, even, or neither.**

$$
\begin{aligned}
f(x) &= x^3 - 3x \\
f(-x) &= (-x)^3 - 3(-x) && \text{Replace } x \text{ with } -x. \\
&= -x^3 + 3x && \text{Simplify.} \\
&= -(x^3 - 3x) && \text{Factor out } -1. \\
&= -f(x) && \text{Substutute.}
\end{aligned}
$$

Therefore, $f(x)$ is odd.

The graph at the right verifies that $f(x)$ is odd. The graph of the function is symmetric with respect to the origin.




**Determine whether each function is *odd*, *even*, or *neither* by graphing or by applying the rules for odd and even functions.**

**1.** $f(x) = 4x^2$

**2.** $f(x) = -7x^4$

**3.** $f(x) = x^7$

**4.** $f(x) = x^3 - x^2$

**5.** $f(x) = 3x^3 + 1$

**6.** $f(x) = x^8 - x^5 - 6$

**7.** $f(x) = -8x^5 - 2x^3 + 6x$

**8.** $f(x) = x^4 - 3x^3 + 2x^2 - 6x + 1$

**9.** $f(x) = x^4 + 3x^2 + 11$

**10.** $f(x) = x^7 - 6x^5 + 2x^3 + x$

**11.** Complete the following definitions: A polynomial function is odd if and only if all the terms are of _____________ degrees. A polynomial function is even if and only if all the terms are of _____________ degrees.

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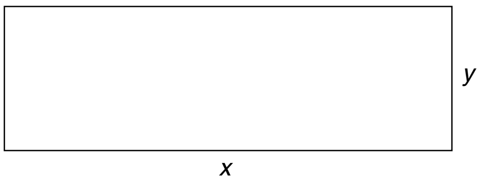
# 7-4 Enrichment

## *Using Maximum Values*

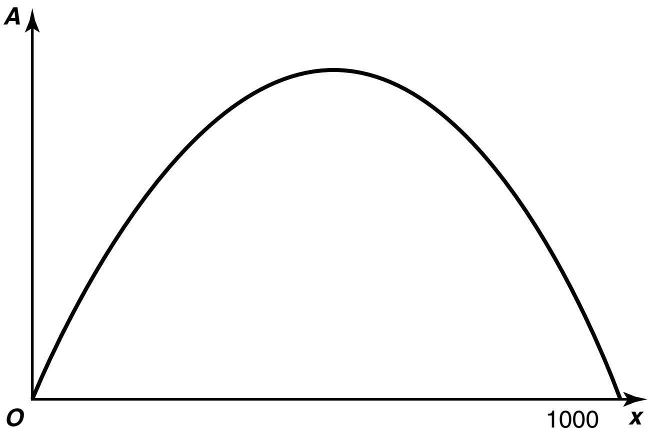
Many times maximum solutions are needed for different situations. For instance, what is the area of the largest rectangular field that can be enclosed with 2000 feet of fencing?

Let $x$ and $y$ denote the length and width of the field, respectively.




Perimeter: $2x + 2y = 2000 \rightarrow y = 1000 - x$
Area: $A = xy = x(1000 - x) = -x^2 + 1000x$

This problem is equivalent to finding the highest point on the graph of $A(x) = -x^2 + 1000x$ shown on the right.




Complete the square for $-x^2 + 1000x$.

$$A = -(x^2 - 1000x + 500^2) + 500^2$$
$$= -(x - 500)^2 + 500^2$$

Because the term $-(x - 500)^2$ is either negative or 0, the greatest value of $A$ is $500^2$. The maximum area enclosed is $500^2$ or 250,000 square feet.

**Solve each problem.**

**1.** Find the area of the largest rectangular garden that can be enclosed by 300 feet of fence.

**2.** A farmer will make a rectangular pen with 100 feet of fence using part of his barn for one side of the pen. What is the largest area he can enclose?

**3.** An area along a straight stone wall is to be fenced. There are 600 meters of fencing available. What is the greatest rectangular area that can be enclosed?

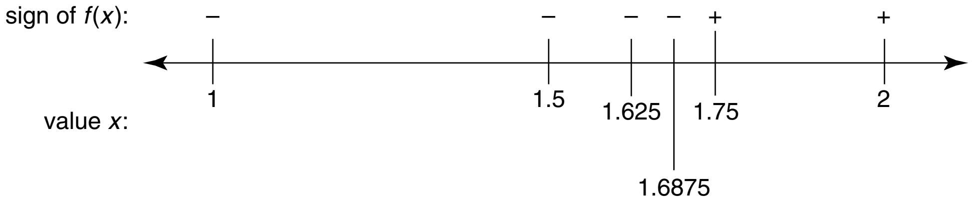
# 7-5 Enrichment

## *The Bisection Method for Approximating Real Zeros*

The **bisection method** can be used to approximate zeros of polynomial functions like $f(x) = x^3 + x^2 - 3x - 3$.

Since $f(1) = -4$ and $f(2) = 3$, there is at least one real zero between 1 and 2. The midpoint of this interval is $\frac{1 + 2}{2} = 1.5$. Since $f(1.5) = -1.875$, the zero is between 1.5 and 2. The midpoint of this interval is $\frac{1.5 + 2}{2} = 1.75$. Since $f(1.75)$ is about 0.172, the zero is between 1.5 and 1.75. The midpoint of this interval is $\frac{1.5 + 1.75}{2} = 1.625$ and $f(1.625)$ is about $-0.94$. The zero is between 1.625 and 1.75. The midpoint of this interval is $\frac{1.625 + 1.75}{2} = 1.6875$. Since $f(1.6875)$ is about $-0.41$, the zero is between 1.6875 and 1.75. Therefore, the zero is 1.7 to the nearest tenth.

The diagram below summarizes the results obtained by the bisection method.




**Using the bisection method, approximate to the nearest tenth the zero between the two integral values of $x$ for each function.**

**1.** $f(x) = x^3 - 4x^2 - 11x + 2$, $f(0) = 2$, $f(1) = -12$

**2.** $f(x) = 2x^4 + x^2 - 15$, $f(1) = -12$, $f(2) = 21$

**3.** $f(x) = x^5 - 2x^3 - 12$, $f(1) = -13$, $f(2) = 4$

**4.** $f(x) = 4x^3 - 2x + 7$, $f(-2) = -21$, $f(-1) = 5$

**5.** $f(x) = 3x^3 - 14x^2 - 27x + 126$, $f(4) = -14$, $f(5) = 16$

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# 7-6 Enrichment

## *Infinite Continued Fractions*

Some infinite expressions are actually equal to real numbers! The infinite continued fraction at the right is one example.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}$$

If you use $x$ to stand for the infinite fraction, then the entire denominator of the first fraction on the right is also equal to $x$. This observation leads to the following equation:

$$x = 1 + \frac{1}{x}$$

**Write a decimal for each continued fraction.**

**1.** $1 + \dfrac{1}{1}$

**2.** $1 + \cfrac{1}{1 + \cfrac{1}{1}}$

**3.** $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$

**4.** $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$

**5.** $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}$

**6.** The more terms you add to the fractions above, the closer their value approaches the value of the infinite continued fraction. What value do the fractions seem to be approaching?

**7.** Rewrite $x = 1 + \dfrac{1}{x}$ as a quadratic equation and solve for $x$.

**8.** Find the value of the following infinite continued fraction.

$$3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}}}$$

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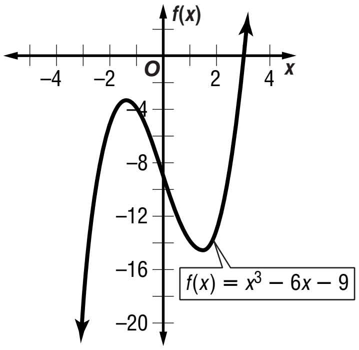
# 7-7 Enrichment

## *Relative Maximum Values*

The graph of $f(x) = x^3 - 6x - 9$ shows a relative maximum value somewhere between $f(-2)$ and $f(-1)$. You can obtain a closer approximation by comparing values such as those shown in the table.

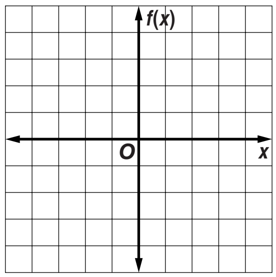
To the nearest tenth a relative maximum value for $f(x)$ is $-3.3$.



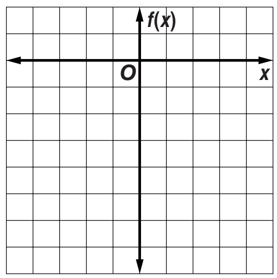

| $x$ | $f(x)$ |
|---|---|
| $-2$ | $-5$ |
| $-1.5$ | $-3.375$ |
| $-1.4$ | $-3.344$ |
| $-1.3$ | $-3.397$ |
| $-1$ | $-4$ |

**Using a calculator to find points, graph each function. To the nearest tenth, find a relative maximum value of the function.**

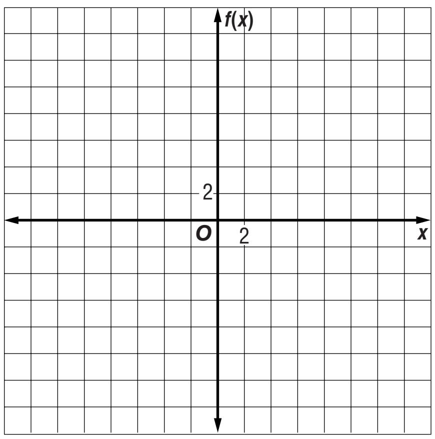
**1.** $f(x) = x(x^2 - 3)$



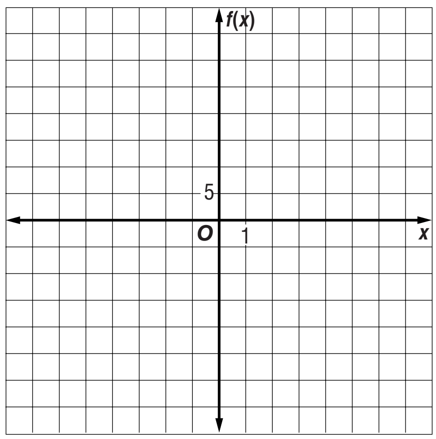

**2.** $f(x) = x^3 - 3x - 3$




**3.** $f(x) = x^3 - 9x - 2$




**4.** $f(x) = x^3 + 2x^2 - 12x - 24$



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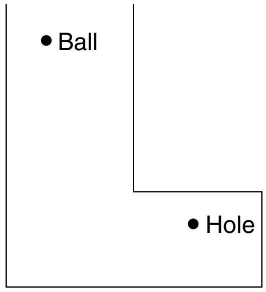
# 7-8 Enrichment

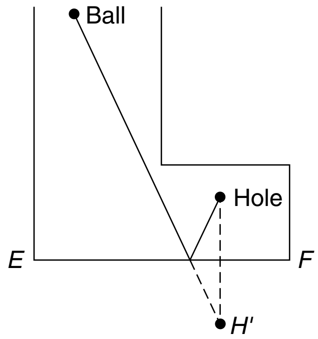
## *Miniature Golf*

In miniature golf, the object of the game is to roll the golf ball into the hole in as few shots as possible. As in the diagram at the right, the hole is often placed so that a direct shot is impossible. Reflections can be used to help determine the direction that the ball should be rolled in order to score a hole-in-one.



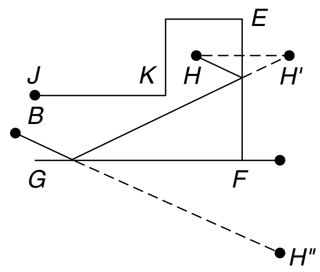

**Example 1** **Using wall $\overline{EF}$, find the path to use to score a hole-in-one.**

Find the reflection image of the "hole" with respect to $\overline{EF}$ and label it $H'$. The intersection of $\overline{BH'}$ with wall $\overline{EF}$ is the point at which the shot should be directed.



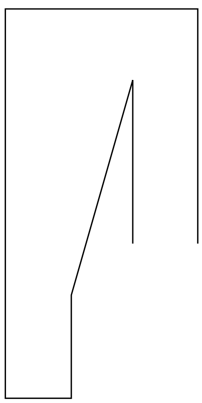

**Example 2** **For the hole at the right, find a path to score a hole-in-one.**

Find the reflection image of $H$ with respect to $\overline{EF}$ and label it $H'$. In this case, $\overline{BH'}$ intersects $\overline{JK}$ before intersecting $\overline{EF}$. Thus, this path cannot be used. To find a usable path, find the reflection image of $H'$ with respect to $\overline{GF}$ and label it $H''$. Now, the intersection of $\overline{BH''}$ with wall $\overline{GF}$ is the point at which the shot should be directed.

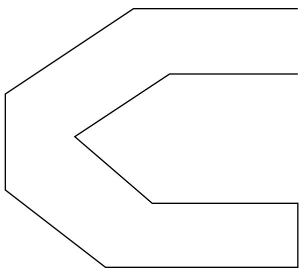



**Copy each figure. Then, use reflections to determine a possible path for a hole-in-one.**

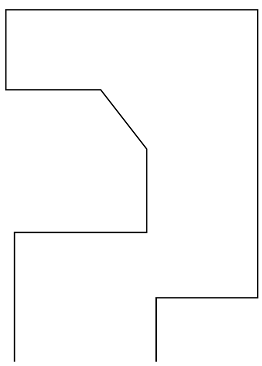
**1.**



**2.**



**3.**



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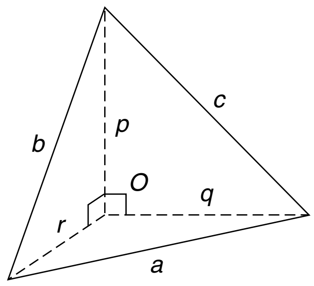
# 7-9 Enrichment

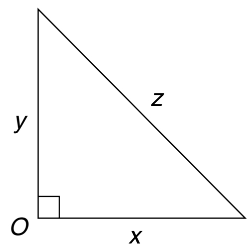
## *Reading Algebra*

If two mathematical problems have basic structural similarities, they are said to be **analogous.** Using analogies is one way of discovering and proving new theorems.

The following numbered sentences discuss a three-dimensional analogy to the Pythagorean theorem.




01 Consider a tetrahedron with three perpendicular faces that meet at vertex *O*.
02 Suppose you want to know how the areas *A, B,* and *C* of the three faces that meet at vertex *O* are related to the area *D* of the face opposite vertex *O*.
03 It is natural to expect a formula analogous to the Pythagorean theorem $z^2 = x^2 + y^2$, which is true for a similar situation in two dimensions.
04 To explore the three-dimensional case, you might guess a formula and then try to prove it.
05 Two reasonable guesses are $D^3 = A^3 + B^3 + C^3$ and $D^2 = A^2 + B^2 + C^2$.




**Refer to the numbered sentences to answer the questions.**

**1.** Use sentence 01 and the top diagram. The prefix *tetra-* means four. Write an informal definition of tetrahedron.

**2.** Use sentence 02 and the top diagram. What are the lengths of the sides of each face of the tetrahedron?

**3.** Rewrite sentence 01 to state a two-dimensional analogue.

**4.** Refer to the top diagram and write expressions for the areas *A, B,* and *C*

**5.** To explore the three-dimensional case, you might begin by expressing *a, b,* and *c* in terms of *p, q,* and *r.* Use the Pythagorean theorem to do this.

**6.** Which guess in sentence 05 seems more likely? Justify your answer.

# 8-1 Enrichment

## *Quadratic Form*

Consider two methods for solving the following equation.

$$(y - 2)^2 - 5(y - 2) + 6 = 0$$

One way to solve the equation is to simplify first, then use factoring.

$$\begin{aligned} y^2 - 4y + 4 - 5y + 10 + 6 &= 0 \\ y^2 - 9y + 20 &= 0 \\ (y - 4)(y - 5) &= 0 \end{aligned}$$

Thus, the solution set is {4, 5}.

Another way to solve the equation is first to replace $y - 2$ by a single variable. This will produce an equation that is easier to solve than the original equation. Let $t = y - 2$ and then solve the new equation.

$$\begin{aligned} (y - 2)^2 - 5(y - 2) + 6 &= 0 \\ t^2 - 5t + 6 &= 0 \\ (t - 2)(t - 3) &= 0 \end{aligned}$$

Thus, $t$ is 2 or 3. Since $t = y - 2$, the solution set of the original equation is {4, 5}.

**Solve each equation using two different methods.**

**1.** $(z + 2)^2 + 8(z + 2) + 7 = 0$

**2.** $(3x - 1)^2 - (3x - 1) - 20 = 0$

**3.** $(2t + 1)^2 - 4(2t + 1) + 3 = 0$

**4.** $(y^2 - 1)^2 - (y^2 - 1) - 2 = 0$

**5.** $(a^2 - 2)^2 - 2(a^2 - 2) - 3 = 0$

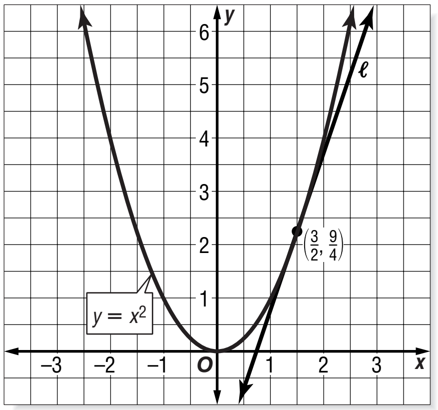
**6.** $(1 + \sqrt{c})^2 + (1 + \sqrt{c}) - 6 = 0$

# 8-2 Enrichment

## *Tangents to Parabolas*

A line that intersects a parabola in exactly one point without crossing the curve is a **tangent** to the parabola. The point where a tangent line touches a parabola is the **point of tangency.** The line perpendicular to a tangent to a parabola at the point of tangency is called the **normal** to the parabola at that point. In the diagram, line $\ell$ is tangent to the parabola that is the graph of $y = x^2$ at $\left(\frac{3}{2}, \frac{9}{4}\right)$. The $x$-axis is tangent to the parabola at $O$, and the $y$-axis is the normal to the parabola at $O$.




**Solve each problem.**

**1.** Find an equation for line $\ell$ in the diagram. *Hint:* A nonvertical line with an equation of the form $y = mx + b$ will be tangent to the graph of $y = x^2$ at $\left(\frac{3}{2}, \frac{9}{4}\right)$ if and only if $\left(\frac{3}{2}, \frac{9}{4}\right)$ is the only pair of numbers that satisfies both $y = x^2$ and $y = mx + b$.

**2.** If $a$ is any real number, then $(a, a^2)$ belongs to the graph of $y = x^2$. Express $m$ and $b$ in terms of $a$ to find an equation of the form $y = mx + b$ for the line that is tangent to the graph of $y = x^2$ at $(a, a^2)$.

**3.** Find an equation for the normal to the graph of $y = x^2$ at $\left(\frac{3}{2}, \frac{9}{4}\right)$.

**4.** If $a$ is a nonzero real number, find an equation for the normal to the graph of $y = x^2$ at $(a, a^2)$.

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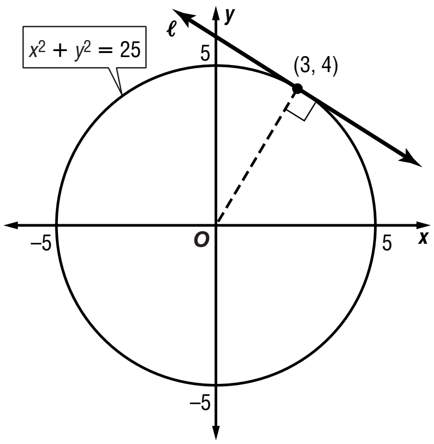
# 8-3 Enrichment

## *Tangents to Circles*

A line that intersects a circle in exactly one point is a **tangent** to the circle. In the diagram, line $\ell$ is tangent to the circle with equation $x^2 + y^2 = 25$ at the point whose coordinates are (3, 4).

A line is tangent to a circle at a point $P$ on the circle if and only if the line is perpendicular to the radius from the center of the circle to point $P$. This fact enables you to find an equation of the tangent to a circle at a point $P$ if you know an equation for the circle and the coordinates of $P$.




**Use the diagram above to solve each problem.**

**1.** What is the slope of the radius to the point with coordinates (3, 4)? What is the slope of the tangent to that point?

**2.** Find an equation of the line $\ell$ that is tangent to the circle at (3, 4).

**3.** If $k$ is a real number between $-5$ and 5, how many points on the circle have $x$-coordinate $k$? State the coordinates of these points in terms of $k$.

**4.** Describe how you can find equations for the tangents to the points you named for Exercise 3.

**5.** Find an equation for the tangent at $(-3, 4)$.

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# 8-4 Enrichment

## *Eccentricity*

In an ellipse, the ratio $\frac{c}{a}$ is called the **eccentricity** and is denoted by the letter $e$. Eccentricity measures the elongation of an ellipse. The closer $e$ is to 0, the more an ellipse looks like a circle. The closer $e$ is to 1, the more elongated it is. Recall that the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ where $a$ is the length of the major axis, and that $c = \sqrt{a^2 - b^2}$.

**Find the eccentricity of each ellipse rounded to the nearest hundredth.**

**1.** $\frac{x^2}{9} + \frac{y^2}{36} = 1$

**2.** $\frac{x^2}{81} + \frac{y^2}{9} = 1$

**3.** $\frac{x^2}{4} + \frac{y^2}{9} = 1$

**4.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$

**5.** $\frac{x^2}{36} + \frac{y^2}{16} = 1$

**6.** $\frac{x^2}{4} + \frac{y^2}{36} = 1$

**7.** Is a circle an ellipse? Explain your reasoning.

**8.** The center of the sun is one focus of Earth's orbit around the sun. The length of the major axis is 186,000,000 miles, and the foci are 3,200,000 miles apart. Find the eccentricity of Earth's orbit.

**9.** An artificial satellite orbiting the earth travels at an altitude that varies between 132 miles and 583 miles above the surface of the earth. If the center of the earth is one focus of its elliptical orbit and the radius of the earth is 3950 miles, what is the eccentricity of the orbit?

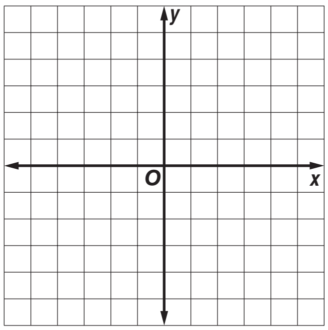
# 8-5 Enrichment

## *Rectangular Hyperbolas*

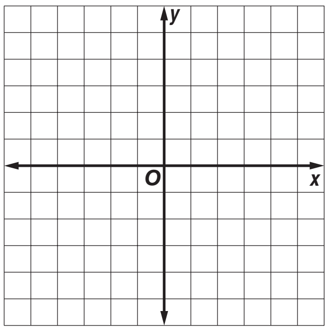
A **rectangular hyperbola** is a hyperbola with perpendicular asymptotes. For example, the graph of $x^2 - y^2 = 1$ is a rectangular hyperbola. A hyperbola with asymptotes that are not perpendicular is called a **nonrectangular hyperbola**. The graphs of equations of the form $xy = c$, where $c$ is a constant, are rectangular hyperbolas.

**Make a table of values and plot points to graph each rectangular hyperbola below. Be sure to consider negative values for the variables.**

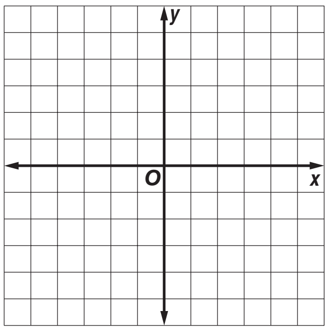
**1.** $xy = -4$



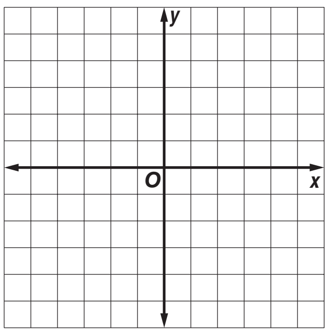

**2.** $xy = 3$




**3.** $xy = -1$




**4.** $xy = 8$




**5.** Make a conjecture about the asymptotes of a rectangular hyperbola where $xy = c$.

# 8-6 Enrichment

## *Loci*

A *locus* (plural, *loci*) is the set of all points, and only those points, that satisfy a given set of conditions. In geometry, figures often are defined as loci. For example, a circle is the locus of points of a plane that are a given distance from a given point. The definition leads naturally to an equation whose graph is the curve described.

Example **Write an equation of the locus of points that are the same distance from (3, 4) and $y = -4$.**

Recognizing that the locus is a parabola with focus (3, 4) and directrix $y = -4$, you can find that $h = 3$, $k = 0$, and $a = 4$ where $(h, k)$ is the vertex and 4 units is the distance from the vertex to both the focus and directrix.

Thus, an equation for the parabola is $y = \frac{1}{16}(x - 3)^2$.

The problem also may be approached analytically as follows:

Let $(x, y)$ be a point of the locus.

The distance from (3, 4) to $(x, y)$ = the distance from $y = -4$ to $(x, y)$.

$$\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-x)^2 + (y-(-4))^2}$$

$$(x-3)^2 + y^2 - 8y + 16 = y^2 + 8y + 16$$

$$(x-3)^2 = 16y$$

$$\frac{1}{16}(x-3)^2 = y$$

**Describe each locus as a geometric figure. Then write an equation for the locus.**

**1.** All points that are the same distance from (0, 5) and (4, 5).

**2.** All points that are 4 units from the origin.

**3.** All points that are the same distance from $(-2, -1)$ and $x = 2$.

**4.** The locus of points such that the sum of the distances from $(-2, 0)$ and $(2, 0)$ is 6.

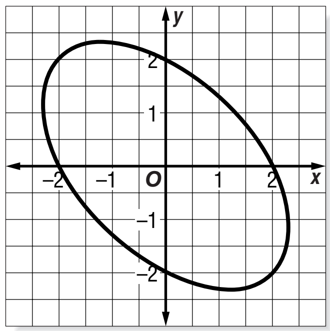
**5.** The locus of points such that the absolute value of the difference of the distances from $(-3, 0)$ and $(3, 0)$ is 2.

# 8-7 Enrichment

## *Graphing Quadratic Equations with xy-Terms*

You can use a graphing calculator to examine graphs of quadratic equations that contain $xy$-terms.

**Example** **Use a graphing calculator to display the graph of $x^2 + xy + y^2 = 4$.**




Solve the equation for $y$ in terms of $x$ by using the quadratic formula.

$y^2 + xy + (x^2 - 4) = 0$

To use the formula, let $a = 1$, $b = x$, and $c = (x^2 - 4)$.

$$y = \frac{-x \pm \sqrt{x^2 - 4(1)(x^2 - 4)}}{2}$$

$$y = \frac{-x \pm \sqrt{16 - 3x^2}}{2}$$

To graph the equation on the graphing calculator, enter the two equations:

$$y = \frac{-x + \sqrt{16 - 3x^2}}{2} \text{ and } y = \frac{-x - \sqrt{16 - 3x^2}}{2}$$

**Use a graphing calculator to graph each equation. State the type of curve each graph represents.**

**1.** $y^2 + xy = 8$

**2.** $x^2 + y^2 - 2xy - x = 0$

**3.** $x^2 - xy + y^2 = 15$

**4.** $x^2 + xy + y^2 = -9$

**5.** $2x^2 - 2xy - y^2 + 4x = 20$

**6.** $x^2 - xy - 2y^2 + 2x + 5y - 3 = 0$

# 9-1 Enrichment

## *Reading Algebra*

In mathematics, the term *group* has a special meaning. The following numbered sentences discuss the idea of group and one interesting example of a group.

01 To be a group, a set of elements and a binary operation must satisfy four conditions: the set must be closed under the operation, the operation must be associative, there must be an identity element, and every element must have an inverse.

02 The following six functions form a group under the operation of composition of functions: $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = \frac{(x-1)}{x}$, $f_5(x) = \frac{x}{(x-1)}$, and $f_6(x) = \frac{1}{(1-x)}$.

03 This group is an example of a noncommutative group. For example, $f_3 \circ f_2 = f_4$, but $f_2 \circ f_3 = f_6$.

04 Some experimentation with this group will show that the identity element is $f_1$.

05 Every element is its own inverse except for $f_4$ and $f_6$, each of which is the inverse of the other.

**Use the paragraph to answer these questions.**

**1.** Explain what it means to say that a set is *closed* under an operation. Is the set of positive integers closed under subtraction?

**2.** Subtraction is a noncommutative operation for the set of integers. Write an informal definition of noncommutative.

**3.** For the set of integers, what is the identity element for the operation of multiplication? Justify your answer.

**4.** Explain how the following statement relates to sentence 05:

$$(f_6 \cdot f_4)(x) = f_6[f_4(x)] = f_6\left(\frac{1}{(1-x)}\right) = \frac{1}{\frac{1-(x-1)}{x}} = x = f_1(x).$$

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# 9-2 Enrichment

## *Superellipses*

The circle and the ellipse are members of an interesting family of curves that were first studied by the French physicist and mathematician Gabriel Lamé (1795–1870). The general equation for the family is

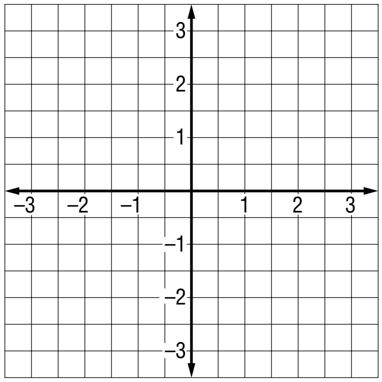
$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1$, with $a \neq 0$, $b \neq 0$, and $n > 0$.

For even values of $n$ greater than 2, the curves are called **superellipses.**

**1.** Consider two curves that are *not* superellipses. Graph each equation on the grid at the right. State the type of curve produced each time.

**a.** $\left|\frac{x}{2}\right|^2 + \left|\frac{y}{2}\right|^2 = 1$

**b.** $\left|\frac{x}{3}\right|^2 + \left|\frac{y}{2}\right|^2 = 1$

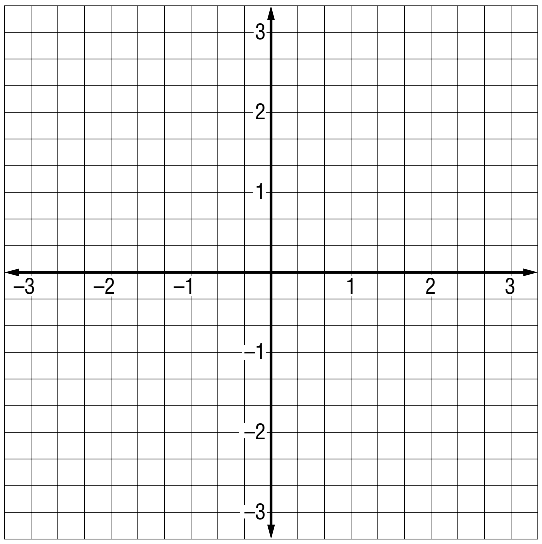



**2.** In each of the following cases you are given values of *a, b,* and $n$ to use in the general equation. Write the resulting equation. Then graph. Sketch each graph on the grid at the right.

**a.** $a = 2, b = 3, n = 4$

**b.** $a = 2, b = 3, n = 6$

**c.** $a = 2, b = 3, n = 8$



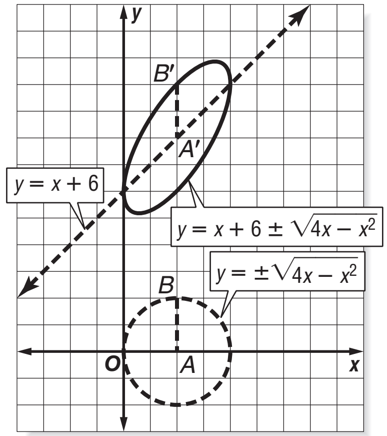

**3.** What shape will the graph of $\left|\frac{x}{2}\right|^n + \left|\frac{y}{2}\right|^n$ approximate for greater and greater even, whole-number values of $n$?

# 9-3 Enrichment

## *Graphing with Addition of y-Coordinates*

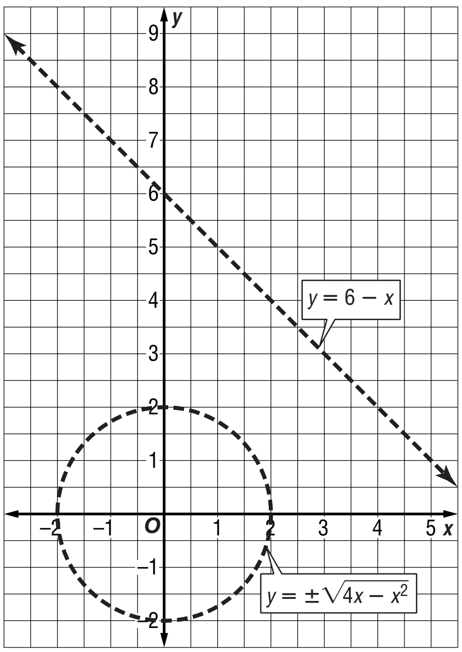
Equations of parabolas, ellipses, and hyperbolas that are "tipped" with respect to the $x$- and $y$-axes are more difficult to graph than the equations you have been studying.

Often, however, you can use the graphs of two simpler equations to graph a more complicated equation. For example, the graph of the ellipse in the diagram at the right is obtained by adding the $y$-coordinate of each point on the circle and the $y$-coordinate of the corresponding point of the line.

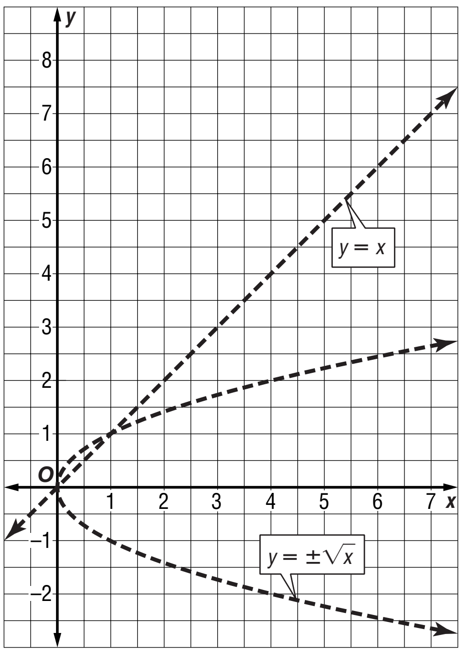



**Graph each equation. State the type of curve for each graph.**

**1.** $y = 6 - x \pm \sqrt{4 - x^2}$




**2.** $y = x \pm \sqrt{x}$




**Use a separate sheet of graph paper to graph these equations. State the type of curve for each graph.**

**3.** $y = 2x \pm \sqrt{7 + 6x - x^2}$

**4.** $y = -2x \pm \sqrt{-2x}$

# 9-4 Enrichment

## *Expansions of Rational Expressions*

Many rational expressions can be transformed into **power series.** A power series is an infinite series of the form $A + Bx + Cx^2 + Dx^3 + \ldots$. The rational expression and the power series normally can be said to have the same values only for certain values of $x$. For example, the following equation holds only for values of $x$ such that $-1 < x < 1$.

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots \text{ for } -1 < x < 1$$

*Example* **Expand $\frac{2 + 3x}{1 + x + x^2}$ in ascending powers of $x$.**

Assume that the expression equals a series of the form $A + Bx + Cx^2 + Dx^3 + \ldots$. Then multiply both sides of the equation by the denominator $1 + x + x^2$.

$$\frac{2 + 3x}{1 + x + x^2} = A + Bx + Cx^2 + Dx^3 + \ldots$$

$$2 + 3x = (1 + x + x^2)(A + Bx + Cx^2 + Dx^3 + \ldots)$$

$$\begin{aligned} 2 + 3x = A + Bx &+ Cx^2 + Dx^3 + \ldots \\ + Ax &+ Bx^2 + Cx^3 + \ldots \\ &+ Ax^2 + Bx^3 + \ldots \end{aligned}$$

$$2 + 3x = A + (B + A)x + (C + B + A)x^2 + (D + C + B)x^3 + \ldots$$

Now, match the coefficients of the polynomials.

$2 = A$

$3 = B + A$

$0 = C + B + A$

$0 = D + C + B + A$

Finally, solve for $A$, $B$, $C$, and $D$ and write the expansion.

$A = 2$, $B = 1$, $C = -3$, and $D = 0$

Therefore, $\frac{2 + 3x}{1 + x + x^2} = 2 + x - 3x^2 + \ldots$

**Expand each rational expression to four terms.**

**1.** $\frac{1 - x}{1 + x + x^2}$

**2.** $\frac{2}{1 - x}$

**3.** $\frac{1}{1 + x}$

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# 9-5 Enrichment

## *Partial Fractions*

It is sometimes an advantage to rewrite a rational expression as the sum of two or more fractions. For example, you might do this in a calculus course while carrying out a procedure called integration.

You can resolve a rational expression into partial fractions if two conditions are met:

(1) The degree of the numerator must be less than the degree of the denominator; and

(2) The factors of the denominator must be known.

**Example** **Resolve $\frac{3}{x^3 + 1}$ into partial fractions.**

The denominator has two factors, a linear factor, $x + 1$, and a quadratic factor, $x^2 - x + 1$. Start by writing the following equation. Notice that the degree of the numerators of each partial fraction is less than its denominator.

$$\frac{3}{x^3 + 1} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$$

Now, multiply both sides of the equation by $x^3 + 1$ to clear the fractions and finish the problem by solving for the coefficients $A$, $B$, and $C$.

$$\frac{3}{x^3 + 1} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$$

$$3 = A(x^2 - x + 1) + (x + 1)(Bx + C)$$

$$3 = Ax^2 - Ax + A + Bx^2 + Cx + Bx + C$$

$$3 = (A + B)x^2 + (B + C - A)x + (A + C)$$

Equating each term, $0x^2 = (A + B)x^2$

$$0x = (B + C - A)x$$

$$3 = (A + C)$$

Therefore, $A = 1, B = -1, C = 2$, and $\frac{3}{x^3 + 1} = \frac{1}{x + 1} + \frac{-x + 2}{x^2 - x + 1}$.

**Resolve each rational expression into partial fractions.**

**1.** $\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}$

**2.** $\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$

**3.** $\frac{4x^3 - x^2 - 3x - 2}{x^2(x + 1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1} + \frac{D}{(x + 1)^2}$

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# 9-6 Enrichment

## *Limits*

Sequences of numbers with a rational expression for the general term often approach some number as a finite limit. For example, the reciprocals of the positive integers approach 0 as $n$ gets larger and larger. This is written using the notation shown below. The symbol $\infty$ stands for infinity and $n \to \infty$ means that $n$ is getting larger and larger, or "$n$ goes to infinity."

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ $\qquad\qquad$ $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

**Example** **Find** $\lim\limits_{n \to \infty} \frac{n^2}{(n+1)^2}$

It is not immediately apparent whether the sequence approaches a limit or not. But notice what happens if we divide the numerator and denominator of the general term by $n^2$.

$$\frac{n^2}{(n+1)^2} = \frac{n^2}{n^2 + 2n + 1}$$

$$= \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}}$$

$$= \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}}$$

The two fractions in the denominator will approach a limit of 0 as $n$ gets very large, so the entire expression approaches a limit of 1.

**Find the following limits.**

**1.** $\lim\limits_{n \to \infty} \frac{n^3 + 5n}{n^4 - 6}$

**2.** $\lim\limits_{n \to \infty} \frac{1 - n}{n^2}$

**3.** $\lim\limits_{n \to \infty} \frac{2(n+1) + 1}{2n + 1}$

**4.** $\lim\limits_{n \to \infty} \frac{2n + 1}{1 - 3n}$

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# 10-1 Enrichment

## *Finding Solutions of $x^y = y^x$*

Perhaps you have noticed that if $x$ and $y$ are interchanged in equations such as $x = y$ and $xy = 1$, the resulting equation is equivalent to the original equation. The same is true of the equation $x^y = y^x$. However, finding solutions of $x^y = y^x$ and drawing its graph is not a simple process.

**Solve each problem. Assume that $x$ and $y$ are positive real numbers.**

**1.** If $a > 0$, will $(a, a)$ be a solution of $x^y = y^x$? Justify your answer.

**2.** If $c > 0$, $d > 0$, and $(c, d)$ is a solution of $x^y = y^x$, will $(d, c)$ also be a solution? Justify your answer.

**3.** Use 2 as a value for $y$ in $x^y = y^x$. The equation becomes $x^2 = 2^x$.

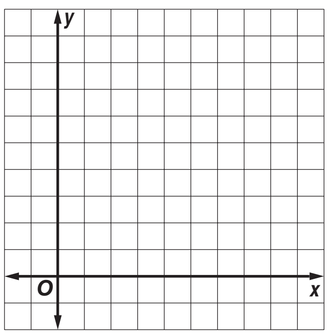
**a.** Find equations for two functions, $f(x)$ and $g(x)$ that you could graph to find the solutions of $x^2 = 2^x$. Then graph the functions on a separate sheet of graph paper.

**b.** Use the graph you drew for part **a** to state two solutions for $x^2 = 2^x$. Then use these solutions to state two solutions for $x^y = y^x$.

**4.** In this exercise, a graphing calculator will be very helpful. Use the technique of Exercise 3 to complete the tables below. Then graph $x^y = y^x$ for positive values of $x$ and $y$. If there are asymptotes, show them in your diagram using dotted lines. Note that in the table, some values of $y$ call for one value of $x$, others call for two.

| *x* | *y* |
|---|---|
| | $\frac{1}{2}$ |
| | $\frac{3}{4}$ |
| | 1 |
| | 2 |
| | 2 |
| | 3 |
| | 3 |

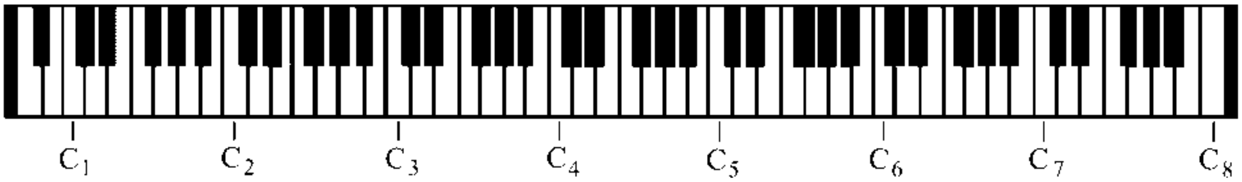
| *x* | *y* |
|---|---|
| | 4 |
| | 4 |
| | 5 |
| | 5 |
| | 8 |
| | 8 |



# 10-2 Enrichment

## *Musical Relationships*

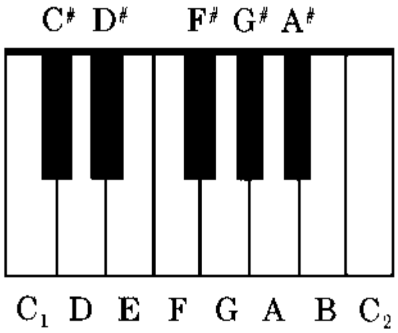
The frequencies of notes in a musical scale that are one octave apart are related by an exponential equation. For the eight C notes on a piano, the equation is $C_n = C_1 2^{n-1}$, where $C_n$ represents the frequency of note $C_n$.




**1.** Find the relationship between $C_1$ and $C_2$.

**2.** Find the relationship between $C_1$ and $C_4$.

The frequencies of consecutive notes are related by a common ratio $r$. The general equation is $f_n = f_1 r^{n-1}$.

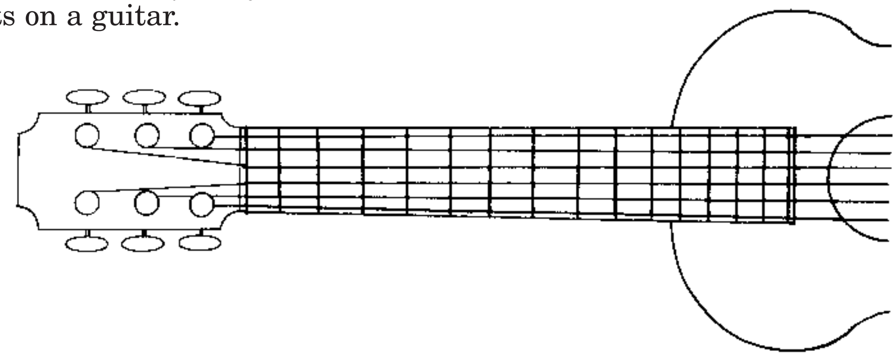



**3.** If the frequency of middle C is 261.6 cycles per second and the frequency of the next higher C is 523.2 cycles per second, find the common ratio $r$. (*Hint:* The two C's are 12 notes apart.) Write the answer as a radical expression.

**4.** Substitute decimal values for $r$ and $f_1$ to find a specific equation for $f_n$.

**5.** Find the frequency of F# above middle C.

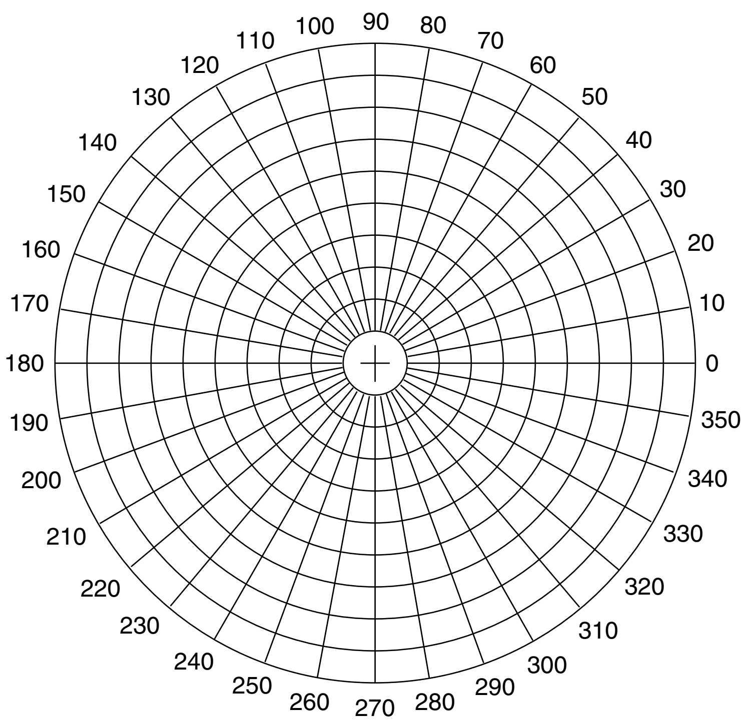
**6.** Frets are a series of ridges placed across the fingerboard of a guitar. They are spaced so that the sound made by pressing a string against one fret has about 1.0595 times the wavelength of the sound made by using the next fret. The general equation is $w_n = w_0(1.0595)^n$. Describe the arrangement of the frets on a guitar.



# 10-3 Enrichment

## *Spirals*

Consider an angle in standard position with its vertex at a point $O$ called the pole. Its initial side is on a coordinatized axis called the *polar axis*. A point $P$ on the terminal side of the angle is named by the *polar coordinates* $(r, \theta)$, where $r$ is the directed distance of the point from $O$ and $\theta$ is the measure of the angle. Graphs in this system may be drawn on polar coordinate paper such as the kind shown below.




**1.** Use a calculator to complete the table for $\log_2 r = \frac{\theta}{120}$.

(*Hint:* To find $\theta$ on a calculator, press 120 $\boxed{\times}$ $\boxed{\text{LOG}}$ $r$ $\boxed{)}$ $\boxed{\div}$ $\boxed{\text{LOG}}$ 2 $\boxed{)}$.)

| *r* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

**2.** Plot the points found in Exercise 1 on the grid above and connect to form a smooth curve.

This type of spiral is called a logarithmic spiral because the angle measures are proportional to the logarithms of the radii.

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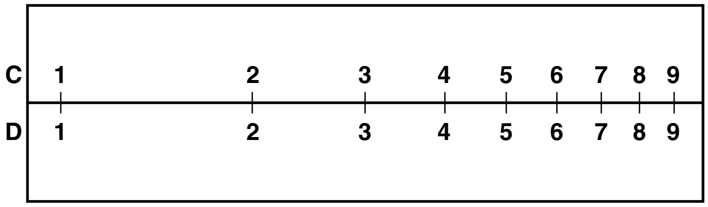
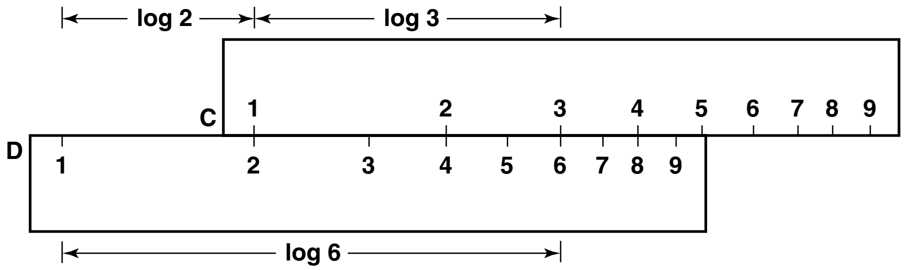
# 10-4 Enrichment

## *The Slide Rule*

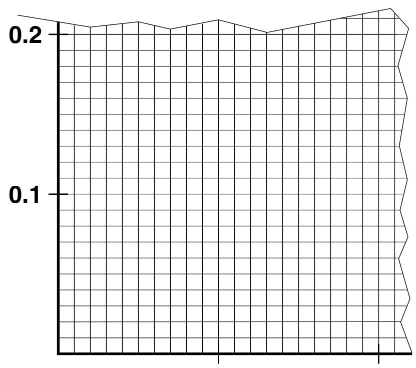
Before the invention of electronic calculators, computations were often performed on a slide rule. A slide rule is based on the idea of logarithms. It has two movable rods labeled with C and D scales. Each of the scales is logarithmic.




To multiply $2 \times 3$ on a slide rule, move the C rod to the right as shown below. You can find $2 \times 3$ by adding log 2 to log 3, and the slide rule adds the lengths for you. The distance you get is 0.778, or the logarithm of 6.



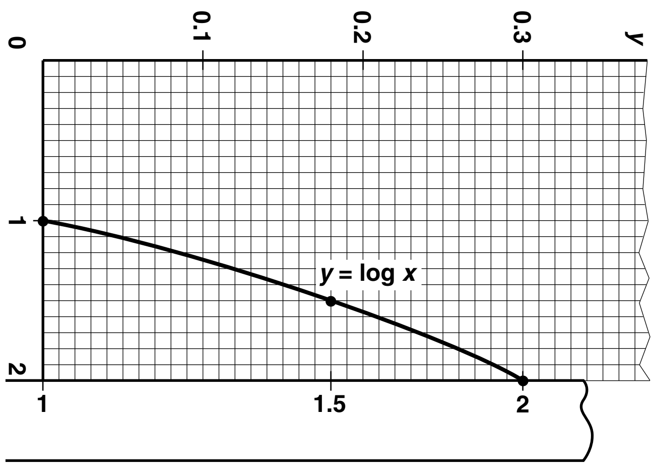

**Follow the steps to make a slide rule.**




**1.** Use graph paper that has small squares, such as 10 squares to the inch. Using the scales shown at the right, plot the curve $y = \log x$ for $x = 1, 1.5$, and the whole numbers from 2 through 10. Make an obvious heavy dot for each point plotted.

**2.** You will need two strips of cardboard. A 5-by-7 index card, cut in half the long way, will work fine. Turn the graph you made in Exercise 1 sideways and use it to mark a logarithmic scale on each of the two strips. The figure shows the mark for 2 being drawn.

**3.** Explain how to use a slide rule to divide 8 by 2.



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# 10-5 Enrichment

## *Approximations for $\pi$ and e*

The following expression can be used to approximate $e$. If greater and greater values of $n$ are used, the value of the expression approximates $e$ more and more closely.

$$\left(1 + \frac{1}{n}\right)^n$$

Another way to approximate $e$ is to use this infinite sum. The greater the value of $n$, the closer the approximation.

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot n} + \ldots$$

In a similar manner, $\pi$ can be approximated using an infinite product discovered by the English mathematician John Wallis (1616–1703).

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \ldots$$

**Solve each problem.**

**1.** Use a calculator with an $e^x$ key to find $e$ to 7 decimal places.

**2.** Use the expression $\left(1 + \frac{1}{n}\right)^n$ to approximate $e$ to 3 decimal places. Use 5, 100, 500, and 7000 as values of $n$.

**3.** Use the infinite sum to approximate $e$ to 3 decimal places. Use the whole numbers from 3 through 6 as values of $n$.

**4.** Which approximation method approaches the value of $e$ more quickly?

**5.** Use a calculator with a $\pi$ key to find $\pi$ to 7 decimal places.

**6.** Use the infinite product to approximate $\pi$ to 3 decimal places. Use the whole numbers from 3 through 6 as values of $n$.

**7.** Does the infinite product give good approximations for $\pi$ quickly?

**8.** Show that $\pi^4 + \pi^5$ is equal to $e^6$ to 4 decimal places.

**9.** Which is larger, $e^{\pi}$ or $\pi^{e}$?

**10.** The expression $x$ reaches a maximum value at $x = e$. Use this fact to prove the inequality you found in Exercise 9.

# 10-6 Enrichment

## *Effective Annual Yield*

When interest is compounded more than once per year, the effective annual yield is higher than the annual interest rate. The effective annual yield, $E$, is the interest rate that would give the same amount of interest if the interest were compounded once per year. If $P$ dollars are invested for one year, the value of the investment at the end of the year is $A = P(1 + E)$. If $P$ dollars are invested for one year at a nominal rate $r$ compounded $n$ times per year, the value of the investment at the end of the year is $A = P\left(1 + \frac{r}{n}\right)^n$. Setting the amounts equal and solving for $E$ will produce a formula for the effective annual yield.

$$P(1 + E) = P\left(1 + \frac{r}{n}\right)^n$$
$$1 + E = \left(1 + \frac{r}{n}\right)^n$$
$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

If compounding is continuous, the value of the investment at the end of one year is $A = Pe^r$. Again set the amounts equal and solve for $E$. A formula for the effective annual yield under continuous compounding is obtained.

$$P(1 + E) = Pe^r$$
$$1 + E = e^r$$
$$E = e^r - 1$$

**Example 1** **Find the effective annual yield of an investment made at 7.5% compounded monthly.**

$r = 0.075$

$n = 12$

$E = \left(1 + \frac{0.075}{12}\right)^{12} - 1 \approx 7.76\%$

**Example 2** **Find the effective annual yield of an investment made at 6.25% compounded continuously.**

$r = 0.0625$

$E = e^{0.0625} - 1 \approx 6.45\%$

**Find the effective annual yield for each investment.**

**1.** 10% compounded quarterly

**2.** 8.5% compounded monthly

**3.** 9.25% compounded continuously

**4.** 7.75% compounded continuously

**5.** 6.5% compounded daily (assume a 365-day year)

**6.** Which investment yields more interest—9% compounded continuously or 9.2% compounded quarterly?

# 11-1 Enrichment

## *Fibonacci Sequence*

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in $n$ months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.
2. Each pair of rabbits produces one pair each month.
3. No rabbits die.

Let $F_n$ represent the number of pairs of rabbits at the end of $n$ months. If you begin with one pair of newborn rabbits, $F_0 = F_1 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1$, or 2. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1$, or 3.

The chart below shows the number of rabbits each month for several months.

| Month | Adult Pairs | Newborn Pairs | Total |
|---|---|---|---|
| $F_0$ | 0 | 1 | 1 |
| $F_1$ | 1 | 0 | 1 |
| $F_2$ | 1 | 1 | 2 |
| $F_3$ | 2 | 1 | 3 |
| $F_4$ | 3 | 2 | 5 |
| $F_5$ | 5 | 3 | 8 |

**Solve.**

**1.** Starting with a single pair of newborn rabbits, how many pairs of rabbits would there be at the end of 12 months?

**2.** Write the first 10 terms of the sequence for which $F_0 = 3, F_1 = 4$, and $F_n = F_{n-2} + F_{n-1}$.

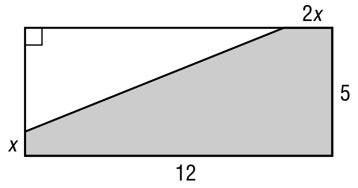
**3.** Write the first 10 terms of the sequence for which $F_0 = 1, F_1 = 5$, $F_n = F_{n-2} + F_{n-1}$.

# 11-2 Enrichment

## *Geometric Puzzlers*

**For the problems on this page, you will need to use the Pythagorean Theorem and the formulas for the area of a triangle and a trapezoid.**

**1.** A rectangle measures 5 by 12 units. The upper left corner is cut off as shown in the diagram.

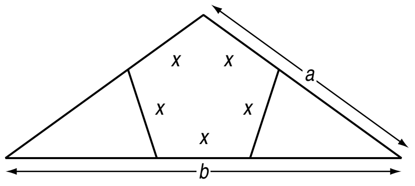



**a.** Find the area $A(x)$ of the shaded pentagon.

**b.** Find $x$ and $2x$ so that $A(x)$ is a maximum. What happens to the cut-off triangle?

**2.** A triangle with sides of lengths $a$, $a$, and $b$ is isosceles. Two triangles are cut off so that the remaining pentagon has five equal sides of length $x$. The value of $x$ can be found using this equation.

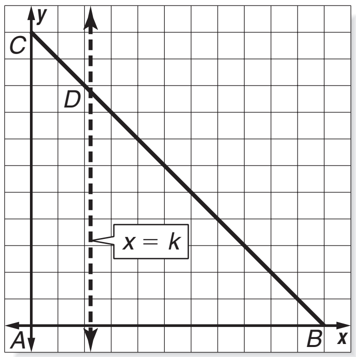
$(2b - a)x^2 + (4a^2 - b^2)(2x - a) = 0$




**a.** Find $x$ when $a = 10$ and $b = 12$.

**b.** Can $a$ be equal to $2b$?

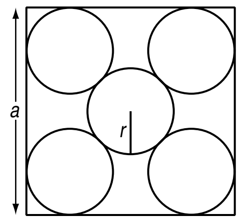
**3.** The coordinates of the vertices of a triangle are $A(0, 0)$, $B(11, 0)$, and $C(0, 11)$. A line $x = k$ cuts the triangle into two regions having equal area.




**a.** What are the coordinates of point $D$?

**b.** Write and solve an equation for finding the value of $k$.

**4.** Inside a square are five circles with the same radius.




**a.** Connect the center of the top left circle to the center of the bottom right circle. Express this length in terms of $r$.

**b.** Draw the square with vertices at the centers of the four outside circles. Express the diagonal of this square in terms of $r$ and $a$.

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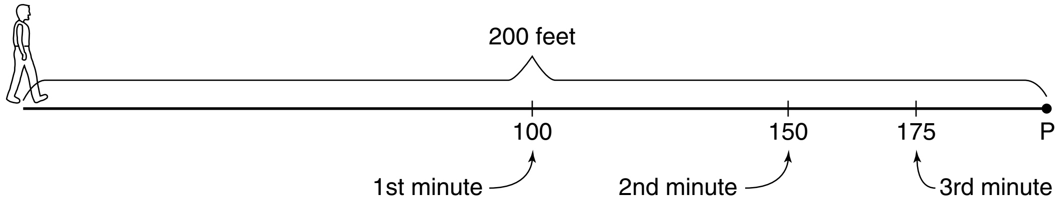
# 11-3 Enrichment

## *Half the Distance*

Suppose you are 200 feet from a fixed point, *P*. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.




An interesting sequence results because according to the problem, you never actually reach the point *P*, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

*Example* **How many minutes are needed to get within 0.1 foot of a point 200 feet away?**

Count the number of times you divide by 2.

**Enter:** 200 [÷] 2 [ENTER] [÷] 2 [ENTER] [÷] 2 [ENTER], and so on

**Result:** 0.0976562

You divided by 2 eleven times. The time needed is 11 minutes.

**Use the method illustrated above to solve each problem.**

**1.** If it is about 2500 miles from Los Angeles to New York, how many minutes would it take to get within 0.1 mile of New York? How far from New York are you at that time?

**2.** If it is 25,000 miles around Earth, how many minutes would it take to get within 0.5 mile of the full distance around Earth? How far short would you be?

**3.** If it is about 250,000 miles from Earth to the Moon, how many minutes would it take to get within 0.5 mile of the Moon? How far from the surface of the Moon would you be?

**4.** If it is about 30,000,000 feet from Honolulu to Miami, how many minutes would it take to get to within 1 foot of Miami? How far from Miami would you be at that time?

**5.** If it is about 93,000,000 miles to the sun, how many minutes would it take to get within 500 miles of the sun? How far from the sun would you be at that time?

# 11-4 Enrichment

## *Annuities*

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that \$30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount $A$. After the third payment, the amount left is

$1.09[1.09A - 30{,}000(1 + 1.09)] - 30{,}000 = 1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$.

The results are summarized in the table below.

| Payment Number | Number of Dollars Left After Payment |
|---|---|
| 1 | $A - 30{,}000$ |
| 2 | $1.09A - 30{,}000(1 + 1.09)$ |
| 3 | $1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$ |

**1.** Use the pattern shown in the table to find the number of dollars left after the fourth payment.

**2.** Find the amount left after the tenth payment.

The amount left after the 14th payment is $1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13})$. However, there should be no money left after the 14th and final payment.

$1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}) = 0$

Notice that $1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}$ is a geometric series where $a_1 = 1$, $a_n = 1.09^{13}$, $n = 14$ and $r = 1.09$.

Using the formula for $S_n$,

$$1 + 1.09 + 1.09^2 + \ldots + 1.09^{13} = \frac{a_1 - a_1r^n}{1 - r} = \frac{1 - 1.09^{14}}{1 - 1.09} = \frac{1 - 1.09^{14}}{-0.09}.$$

**3.** Show that when you solve for $A$ you get $A = \frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$.

Therefore, to provide \$30,000 for 14 years where the annual interest rate is 9%, you need $\frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$ dollars.

**4.** Use a calculator to find the value of $A$ in problem 3.

In general, if you wish to provide $P$ dollars for each of $n$ years at an annual rate of $r$%, you need $A$ dollars where

$$\left(1 + \frac{r}{100}\right)^{n-1}A - P\left[1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2 + \ldots + \left(1 + \frac{r}{100}\right)^{n-1}\right] = 0.$$

You can solve this equation for $A$, given $P$, $n$, and $r$.

# 11-5 Enrichment

## *Convergence and Divergence*

Convergence and divergence are terms that relate to the existence of a sum of an infinite series. If a sum exists, the series is convergent. If not, the series is divergent. Consider the series $12 + 3 + \frac{3}{4} + \frac{3}{16} + \ldots$. This is a geometric series with $r = \frac{1}{4}$. The sum is given by the formula $S = \frac{a_1}{1 - r}$. Thus, the sum is $12 \div \frac{3}{4}$ or 16. This series is convergent since a sum exists. Notice that the first two terms have a sum of 15. As more terms are added, the sum comes closer (or converges) to 16.

Recall that a geometric series has a sum if and only if $-1 < r < 1$. Thus, a geometric series is convergent if $r$ is between $-1$ and 1, and divergent if $r$ has another value. An infinite arithmetic series cannot have a sum unless all of the terms are equal to zero.

**Example** **Determine whether each series is convergent or divergent.**

**a.** $\mathbf{2 + 5 + 8 + 11 + \ldots}$ divergent

**b.** $\mathbf{-2 + 4 + (-8) + 16 + \ldots}$ divergent

**c.** $\mathbf{16 + 8 + 4 + 2 + \ldots}$ convergent

**Determine whether each series is convergent or divergent. If the series is convergent, find the sum.**

**1.** $5 + 10 + 15 + 20 + \ldots$

**2.** $16 + 8 + 4 + 2 + \ldots$

**3.** $1 + 0.1 + 0.01 + 0.001 + \ldots$

**4.** $4 + 2 + 0 - 2 - \ldots$

**5.** $2 - 4 + 8 - 16 + \ldots$

**6.** $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \ldots$

**7.** $4 + 2.4 + 1.44 + 0.864 + \ldots$

**8.** $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 + \ldots$

**9.** $-\frac{5}{3} + \frac{10}{9} - \frac{20}{27} + \frac{40}{81} - \ldots$

**10.** $48 + 12 + 3 + \frac{3}{4} + \ldots$

***Bonus:*** Is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ convergent or divergent?

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# 11-6 Enrichment

## *Continued Fractions*

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$$

**Example 1** **Evaluate the continued fraction above. Start at the bottom and work your way up.**

**Step 1:** $4 + \frac{1}{5} = \frac{20}{5} + \frac{1}{5} = \frac{21}{5}$

**Step 2:** $\frac{1}{\frac{21}{5}} = \frac{5}{21}$

**Step 3:** $3 + \frac{5}{21} = \frac{63}{21} + \frac{5}{21} = \frac{68}{21}$

**Step 4:** $\frac{1}{\frac{68}{21}} = \frac{21}{68}$

**Step 5:** $2 + \frac{21}{68} = 2\frac{21}{68}$

**Example 2** **Change $\frac{25}{11}$ into a continued fraction.**

Follow the steps.

**Step 1:** $\frac{25}{11} = \frac{22}{11} + \frac{3}{11} = 2 + \frac{3}{11}$

**Step 2:** $\frac{3}{11} = \frac{1}{\frac{11}{3}}$

**Step 3:** $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3}$

**Step 4:** $\frac{2}{3} = \frac{1}{\frac{3}{2}}$

**Step 5:** $\frac{3}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2}$

Stop, because the numerator is 1.

Thus, $\frac{25}{11}$ can be written as $2 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}$

**Evaluate each continued fraction.**

**1.** $1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{3}}}}$

**2.** $0 + \cfrac{1}{6 + \cfrac{1}{4 + \frac{1}{2}}}$

**3.** $2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \frac{1}{10}}}}$

**4.** $5 + \cfrac{1}{7 + \cfrac{1}{9 + \frac{1}{11}}}$

**Change each fraction into a continued fraction.**

**5.** $\frac{75}{31}$

**6.** $\frac{29}{8}$

**7.** $\frac{13}{19}$

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# 11-7 Enrichment

## *Patterns in Pascal's Triangle*

You have learned that the coefficients in the expansion of $(x + y)^n$ yield a number pyramid called **Pascal's triangle.**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Row 1 → | | | | | | | 1 | | | | | | |
| Row 2 → | | | | | | 1 | | 1 | | | | | |
| Row 3 → | | | | | 1 | | 2 | | 1 | | | | |
| Row 4 → | | | | 1 | | 3 | | 3 | | 1 | | | |
| Row 5 → | | | 1 | | 4 | | 6 | | 4 | | 1 | | |
| Row 6 → | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | |
| Row 7 → | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 |

As many rows can be added to the bottom of the pyramid as you please.

This activity explores some of the interesting properties of this famous number pyramid.

**1.** Pick a row of Pascal's triangle.

**a.** What is the sum of all the numbers in all the rows *above* the row you picked?

**b.** What is the sum of all the numbers in the row you picked?

**c.** How are your answers for parts **a** and **b** related?

**d.** Repeat parts **a** through **c** for at least three more rows of Pascal's triangle. What generalization seems to be true?

**e.** See if you can prove your generalization.

**2.** Pick any row of Pascal's triangle that comes after the first.

**a.** Starting at the left end of the row, add the first number, the third number, the fifth number, and so on. State the sum.

**b.** In the same row, add the second number, the fourth number, and so on. State the sum.

**c.** How do the sums in parts **a** and **b** compare?

**d.** Repeat parts **a** through **c** for at least three other rows of Pascal's triangle. What generalization seems to be true?

# 11-8 Enrichment

## *Proof by Induction*

Mathematical induction is a useful tool when you want to prove that a statement is true for all natural numbers.

The three steps in using induction are:

**1.** Prove that the statement is true for $n = 1$.

**2.** Prove that if the statement is true for the natural number $n$, it must also be true for $n + 1$.

**3.** Conclude that the statement is true for all natural numbers.

**Follow the steps to complete each proof.**

**Theorem A:** The sum of the first $n$ odd natural numbers is equal to $n^2$.

**1.** Show that the theorem is true for $n = 1$.

**2.** Suppose $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Show that $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$.

**3.** Summarize the results of problems 1 and 2.

**Theorem B:** Show that $a^n - b^n$ is exactly divisible by $a - b$ for $n$ equal to 1, 2, 3, and all natural numbers.

**4.** Show that the theorem is true for $n = 1$.

**5.** The expression $a^{n+1} - b^{n+1}$ can be rewritten as $a(a^n - b^n) + b^n(a - b)$. Verify that this is true.

**6.** Suppose $a - b$ is a factor of $a^n - b^n$. Use the result in problem 5 to show that $a - b$ must then also be a factor of $a^{n+1} - b^{n+1}$.

**7.** Summarize the results of problems 4 through 6.

 

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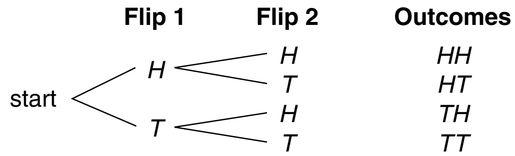
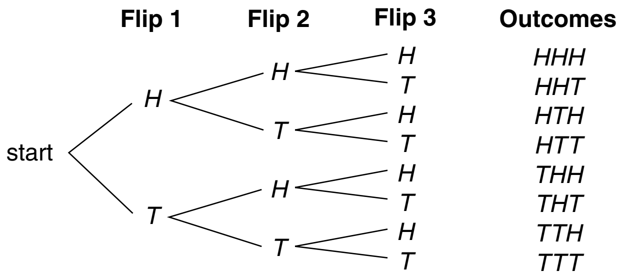
# 12-1 Enrichment

## *Tree Diagrams and the Power Rule*

If you flip a coin once, there are two possible outcomes: heads showing ($H$) or tails showing ($T$). The tree diagram to the right shows the four ($2^2$) possible outcomes if you flip a coin twice.

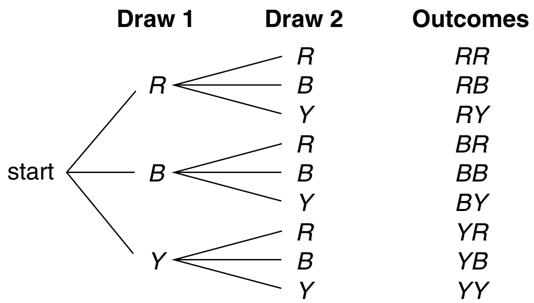



**Example 1** **Draw a tree diagram to show all the possible outcomes for flipping a coin three times. List the outcomes.**




There are eight ($2^3$) possible outcomes. With each extra flip, the number of outcomes doubles. With 4 flips, there would be sixteen ($2^4$) outcomes.

**Example 2** **In a cup there are a red, a blue, and a yellow marble. How many possible outcomes are there if you draw one marble at random, replace it, and then draw another?**




There are nine ($3^2$) possible outcomes.

The Power Rule for the number of outcomes states that if an experiment is repeated $n$ times, and if there are $b$ possible outcomes each time, there are $b^n$ total possible outcomes.

**Find the total number of possible outcomes for each experiment. Use tree diagrams to help you.**

**1.** flipping a coin 5 times

**2.** doing the marble experiment 6 times

**3.** flipping a coin 8 times

**4.** rolling a 6-sided die 2 times

**5.** rolling a 6-sided die 3 times

**6.** rolling a 4-sided die 2 times

**7.** rolling a 4-sided die 3 times

**8.** rolling a 12-sided die 2 times

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# 12-2 Enrichment

## *Combinations and Pascal's Triangle*

Pascal's triangle is a special array of numbers invented by Blaise Pascal (1623–1662). The values in Pascal's triangle can be found using the combinations shown below.

**1.** Evaluate the expression in each cell of the triangle.

| | | | | | |
|---|---|---|---|---|---|
| C(1,0) | C(1,1) | | | | |
| C(2,0) | C(2,1) | C(2,2) | | | |
| C(3,0) | C(3,1) | C(3,2) | C(3,3) | | |
| C(4,0) | C(4,1) | C(4,2) | C(4,3) | C(4,4) | |
| C(5,0) | C(5,1) | C(5,2) | C(5,3) | C(5,4) | C(5,5) |

**2.** The pattern shows the relationship between $C(n, r)$ and Pascal's triangle. In general, it is true that $C(n, r) + C(n, r + 1) = C(n + 1, r + 1)$. Complete the proof of this property. In each step, the denominator has been given.

$$
\begin{aligned}
C(n, r) + C(n, r + 1) &= \frac{\qquad}{r!(n - r)!} + \frac{\qquad}{(r + 1)!(n - r - 1)!} \\
&= \frac{\qquad}{r!(n - r)!(r + 1)} + \frac{\qquad}{(r + 1)!(n - r - 1)!(n - r)} \\
&= \frac{\qquad}{(r + 1)!(n - r)!} + \frac{\qquad}{(r + 1)!(n - r)!} \\
&= \frac{\qquad}{(r + 1)!(n - r)!} \\
&= \frac{\qquad}{(r + 1)!(n - r)!} \\
&= \frac{\qquad}{(r + 1)!(n - r)!} \\
&= \frac{\qquad}{(r + 1)![(n + 1) - (r + 1)]!} \\
&= C(n + 1, r + 1)
\end{aligned}
$$

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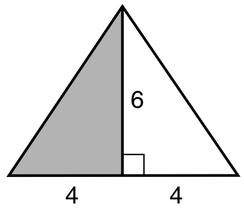
# 12-3 Enrichment

## *Geometric Probability*

If a dart, thrown at random, hits the triangular board shown at the right, what is the chance that it will hit the shaded region? This chance, also called a probability, can be determined by comparing the area of the shaded region to the area of the board. This ratio indicates what fraction of the tosses should hit in the shaded region.




$$\frac{\text{area of shaded region}}{\text{area of triangular board}} = \frac{\frac{1}{2}(4)(6)}{\frac{1}{2}(8)(6)}$$

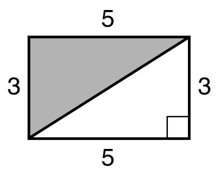
$$= \frac{12}{24} \text{ or } \frac{1}{2}$$

In general, if $S$ is a subregion of some region $R$, then the probability, $P(S)$, that a point, chosen at random, belongs to subregion $S$ is given by the following.

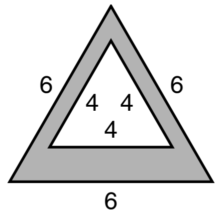
$$P(S) = \frac{\text{area of subregion } S}{\text{are of region } R}$$

**Find the probability that a point, chosen at random, belongs to the shaded subregions of the following regions.**

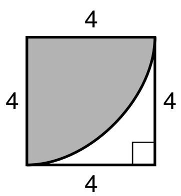
**1.**



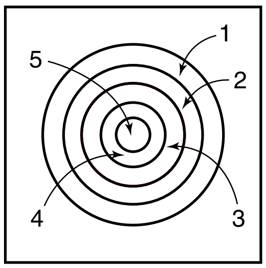

**2.**




**3.**




**The dart board shown at the right has 5 concentric circles whose centers are also the center of the square board. Each side of the board is 38 cm, and the radii of the circles are 2 cm, 5 cm, 8 cm, 11 cm, and 14 cm. A dart hitting within one of the circular regions scores the number of points indicated on the board, while a hit anywhere else scores 0 points. If a dart, thrown at random, hits the board, find the probability of scoring the indicated number of points.**




**4.** 0 points

**5.** 1 point

**6.** 2 points

**7.** 3 points

**8.** 4 points

**9.** 5 points

# 12-4 Enrichment

## *Conditional Probability*

Suppose a pair of dice is thrown. It is known that the sum is greater than seven. Find the probability that the dice match.

The probability of an event given the occurrence of another event is called *conditional probability.* The conditional probability of event $A$, the dice match, given event $B$, their sum is greater than seven, is denoted $P(A/B)$.

There are 15 sums greater than seven and there are 36 possible pairs altogether.

$P(B) = \frac{15}{36}$

There are three matching pairs greater than seven.

$P(A \text{ and } B) = \frac{3}{36}$

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(A/B) = \frac{\frac{3}{36}}{\frac{15}{36}} \text{ or } \frac{1}{5}$$

The conditional probability is $\frac{1}{5}$.

**A card is drawn from a standard deck of 52 and is found to be red. Given that event, find each of the following probabilities.**

**1.** $P$(heart)

**2.** $P$(ace)

**3.** $P$(face card)

**4.** $P$(jack or ten)

**5.** $P$(six of spades)

**6.** $P$(six of hearts)

**A sports survey taken at Stirers High School shows that 48% of the respondents liked soccer, 66% liked basketball, and 38% liked hockey. Also, 30% liked soccer and basketball, 22% liked basketball and hockey and 28% liked soccer and hockey. Finally, 12% liked all three sports. Find each of the following probabilities.**

**7.** The probability Meg likes soccer if she likes basketball.

**8.** The probability Biff likes basketball if he likes soccer.

**9.** The probability Muffy likes hockey if she likes basketball.

**10.** The probability Greg likes hockey and basketball if he likes soccer.

# 12-5 Enrichment

## *Probability and Tic-Tac-Toe*

What would be the chances of winning at tic-tac-toe if it were turned into a game of pure chance? To find out, the nine cells of the tic-tac-toe board are numbered from 1 to 9 and nine chips (also numbered from 1 to 9) are put into a bag. Player A draws a chip at random and enters an *X* in the corresponding cell. Player B does the same and enters an *O*.

To solve the problem, assume that both players draw all their chips without looking and all *X* and *O* entries are made at the same time. There are four possible outcomes: a draw, A wins, B wins, and either A or B can win.

There are 16 arrangements that result in a draw. Reflections and rotations must be counted as shown below.

| | | |
|---|---|---|
| o x o | x o x | o o x |
| x o x 4 | o o x 4 | x x o 8 |
| x o x | x x o | o x x |

There are 36 arrangements in which either player may win because both players have winning triples.

| | | | | | |
|---|---|---|---|---|---|
| x x x | x x x | x o x | x x x | x x x | x x o |
| o o o 4 | x o x 4 | x x x 4 | x x o 8 | o o o 8 | x x x 8 |
| x o x | o o o | o o o | o o o | x x o | o o o |

In these 36 cases, A's chances of winning are $\frac{13}{40}$.

**1.** Find the 12 arrangements in which B wins and A cannot.

**2.** Below are 12 of the arrangements in which A wins and B cannot. Write the numbers to show the reflections and rotations for each arrangement. What is the total number?

| | | | | | |
|---|---|---|---|---|---|
| o x o | x o x | x x x | x x x | x o o | x o o |
| x x x | o x o | x o o | o x o | x x x | x x o |
| o x o | x o x | x o o | o x o | o o x | o o x |

| | | | | | |
|---|---|---|---|---|---|
| x x o | x x x | x x x | x x x | x o o | x x o |
| o x x | o x o | x o o | x o o | x x x | o x o |
| o o x | o o x | o x o | o o x | o x o | x o x |

**3.** There are $\frac{9!}{(5!4!)}$ different and equally probable distributions. Complete the chart to find the probability for a draw or for A or B to win.

| | | | |
|---|---|---|---|
| Draw: $\frac{16}{126}$ | | | = ________ |
| A wins: ________ | + | $\frac{13}{40}\left(\frac{36}{126}\right)$ | = ________ |
| B wins: ________ | + | ________ | = ________ |

# 12-6 Enrichment

## *Probabilities in Genetics*

**Genes** are the units which transmit hereditary traits. The possible forms which a gene may take, **dominant** and **recessive,** are called **alleles.** A particular trait is determined by two alleles, one from the female parent and one from the male parent. If an organism has the trait which is dominant, it may have either two dominant alleles or one dominant and one recessive allele. If the organism has the trait which is recessive, it must have two recessive alleles.

*Example* **Consider a plant in which tall stems, *T*, are dominant to short stems, *t*. What is the probability of obtaining a long-stemmed plant if two long-stemmed plants both with the genetic formula *Tt* are crossed?**

A *Punnett square* is a chart used to determine the possible combinations of characteristics among offspring.

| | *T* | *t* |
|---|---|---|
| *T* | *TT* | *Tt* |
| *t* | *Tt* | *tt* |

$$\begin{array}{r} 3 \text{ tall-stemmed} \\ +\ 1 \text{ short-stemmed} \\ \hline 4 \text{ total} \end{array}$$

Thus, the probability is $\frac{3}{4}$.

**In a certain plant, red flowers, *R*, are dominant to white flowers, *r*. If a white-flowered plant, *rr* is crossed with a red-flowered plant, *Rr*, find the probability of each of the following.**

**1.** white-flowered plant

**2.** red-flowered plant

**In a certain plant, tall, *T*, is dominant to short, *t*, and green pods, *G*, are dominant to yellow pods, *g*. Plants with the genetic formulas *TtGg* and *TTGg* are crossed. Find the probability of each of the following.**

**3.** tall plant with green pods

**4.** tall plant with yellow pods

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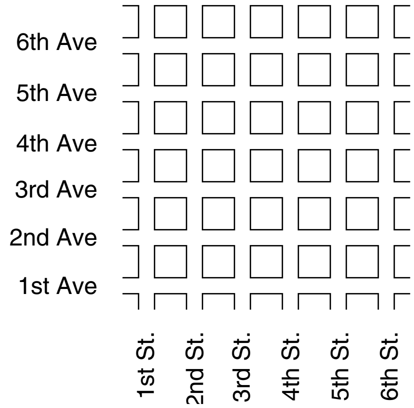
# 12-7 Enrichment

## *Street Networks: Finding All Possible Routes*




A section of a city is laid out in square blocks. Going north from the intersection of First Avenue and First Street, the avenues are 1st, 2nd, 3rd, and so on. Going east, the streets are numbered in the same way.

Factorials can be used to find the number, $r(e, n)$, of different routes between two intersections. The formula is shown below.

$$r(e, n) = \frac{[(e - 1) + (n - 1)]!}{(e - 1)!(n - 1)!}$$

The number of streets going east is $e$; the number of avenues going north is $n$.

The following problems examine the possible routes from one location to another. Assume that you never use a route that is unnecessarily long. Assume that $e \geq 1$ and $n \geq 1$.

**Solve each problem.**

**1.** List all the possible routes from 1st Street and 1st Avenue to 4th Street and 3rd Avenue. Use ordered pairs to show the routes, with street numbers first, and avenue numbers second. For example, each route starts at (1, 1) and ends at (4, 3).

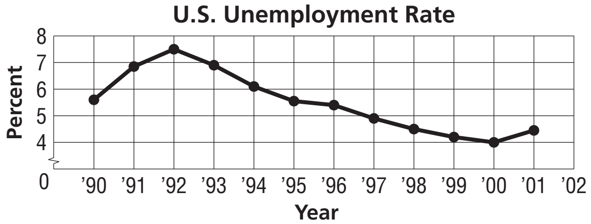
**2.** Use the formula to compute the number of routes from (1, 1) to (4, 3). There are 4 streets going east and 3 avenues going north.

**3.** Find the number of routes from 1st Street and 1st Avenue to 7th Street and 6th Avenue.

# 12-8 Enrichment

## *Misuses of Statistics*

Statistics can be misleading. Graphs for a set of data can look very different from one another. Compare the following graphs.




**Source:** U.S. Department of Labor


**Source:** U.S. Department of Labor

Notice that the two graphs show the same data, but the spacing in the vertical and horizontal scales differs. Scales can be cramped or spread out to make a graph that gives a certain impression. Which graph would you use to give the impression that the unemployment rate dropped dramatically from 1990 to 2000?

Suppose that a car company claims, "75% of people surveyed say that our car is better than the competition." If four people were asked which car they preferred and 75% agreed, how many people thought that *Our Car* was better?

The advertisement was misleading in other ways as well. For example, who was surveyed—were the people company employees, or impartial buyers?

**Suppose an advertiser claims that 90% of all of one brand of car sold in the last 10 years are still on the road.**

**1.** If 10,000 cars were sold, how many are still on the road?

**2.** If 1000 cars were sold, how many are still on the road?

**3.** Find an example to show how you think averages could be used in a misleading way.

**4.** A survey of a large sample of people who own small computers revealed that 85% of the people thought the instruction manuals should be better written. A manufacturer of small computers claimed that it surveyed many of the same people and found that all of them liked their manuals. Discuss the possible discrepancy in the results.

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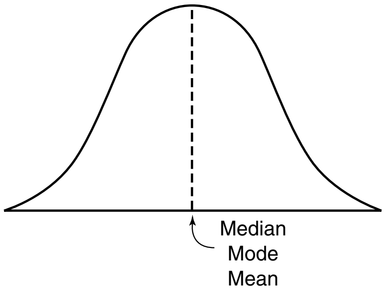
# 12-9 Enrichment

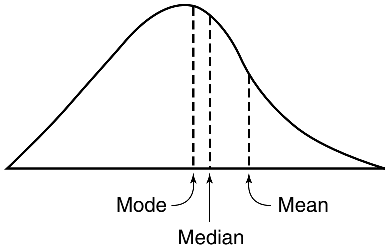
## *Shapes of Distribution Curves*

Graphs of frequency distributions can be described as either symmetric or skewed.

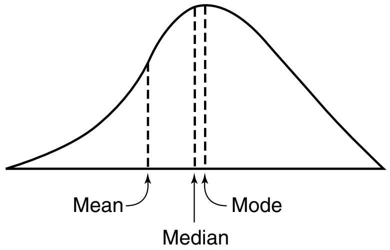
**Symmetric**




**Skewed to the Right**




**Skewed to the Left**

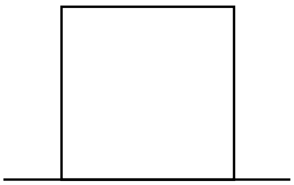



In a distribution skewed to the right, there are a larger number of high values. The long “tail” extends to the right.

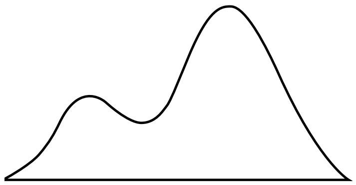
In a distribution skewed to the left, there are a larger number of low values. The “tail” extends to the left.

**For each of the following, state whether the distribution is symmetric or skewed. If it is skewed, tell whether it is skewed to the right or to the left.**

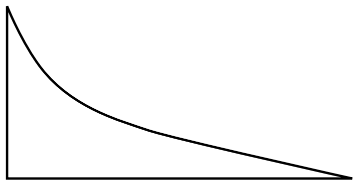
**1.**



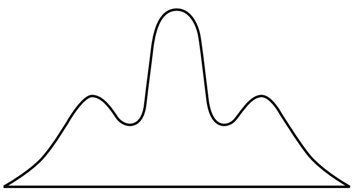
**2.**



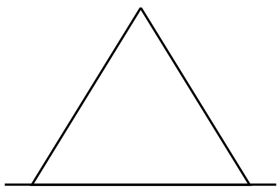
**3.**



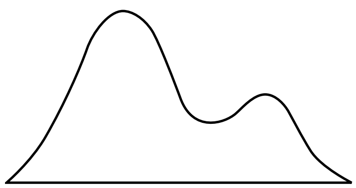
**4.**



**5.**



**6.**



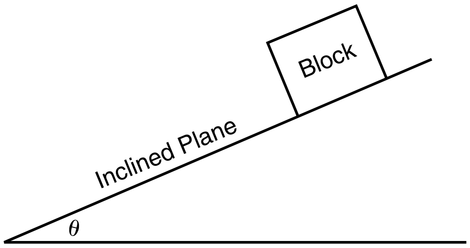
**A vertical line above the median divides the area under a frequency curve in half.**

**7.** Where is the median in a symmetric distribution?

**8.** Where is the median in a skewed distribution?

# 13-1 Enrichment

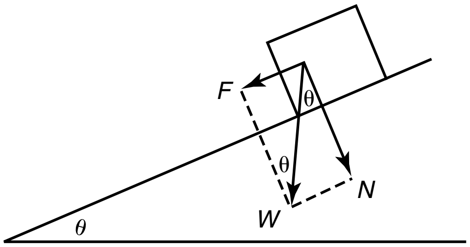
## *The Angle of Repose*




Suppose you place a block of wood on an inclined plane, as shown at the right. If the angle, $\theta$, at which the plane is inclined from the horizontal is very small, the block will not move. If you increase the angle, the block will eventually overcome the force of friction and start to slide down the plane.

At the instant the block begins to slide, the angle formed by the plane is called the angle of friction, or the angle of repose.

For situations in which the block and plane are smooth but unlubricated, the angle of repose depends *only* on the types of materials in the block and the plane. The angle is independent of the area of contact between the two surfaces and of the weight of the block.




The drawing at the right shows how to use vectors to find a coefficient of friction. This coefficient varies with different materials and is denoted by the Greek leter mu, $\mu$.

$$F = W \sin \theta \qquad N = W \cos$$
$$F = \mu N$$
$$\mu = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

**Solve each problem.**

| Material | Coefficient of Friction $\mu$ |
|---|---|
| Wood on wood | 0.5 |
| Wood on stone | 0.5 |
| Rubber tire on dry concrete | 1.0 |
| Rubber tire on wet concrete | 0.7 |

**1.** A wooden chute is built so that wooden crates can slide down into the basement of a store. What angle should the chute make in order for the crates to slide down at a constant speed?

**2.** Will a 100-pound wooden crate slide down a stone ramp that makes an angle of 20° with the horizontal? Explain your answer.

**3.** If you increase the weight of the crate in Exercise 2 to 300 pounds, does it change your answer?

**4.** A car with rubber tires is being driven on dry concrete pavement. If the car tires spin without traction on a hill, how steep is the hill?

**5.** For Exercise 4, does it make a difference if it starts to rain? Explain your answer.

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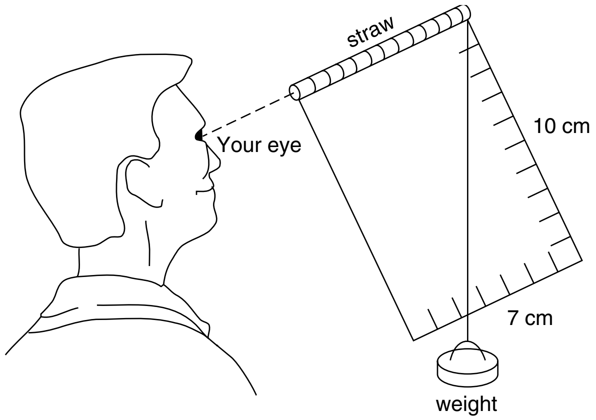
# 13-2 Enrichment

## *Making and Using a Hypsometer*

A **hypsometer** is a device that can be used to measure the height of an object. To construct your own hypsometer, you will need a rectangular piece of heavy cardboard that is at least 7 cm by 10 cm, a straw, transparent tape, a string about 20 cm long, and a small weight that can be attached to the string.

Mark off 1-cm increments along one short side and one long side of the cardboard. Tape the straw to the other short side. Then attach the weight to one end of the string, and attach the other end of the string to one corner of the cardboard, as shown in the figure below. The diagram below shows how your hypsometer should look.




To use the hypsometer, you will need to measure the distance from the base of the object whose height you are finding to where you stand when you use the hypsometer.

Sight the top of the object through the straw. Note where the free-hanging string crosses the bottom scale. Then use similar triangles to find the height of the object.

1. Draw a diagram to illustrate how you can use similar triangles and the hypsometer to find the height of a tall object.

**Use your hypsometer to find the height of each of the following.**

2. your school's flagpole

3. a tree on your school's property

4. the highest point on the front wall of your school building

5. the goal posts on a football field

6. the hoop on a basketball court

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# 13-3 Enrichment

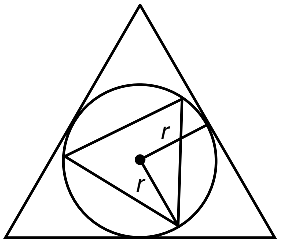
## *Areas of Polygons and Circles*

A regular polygon has sides of equal length and angles of equal measure. A regular polygon can be inscribed in or circumscribed about a circle. For $n$-sided regular polygons, the following area formulas can be used.

Area of circle $A_C = \pi r^2$

Area of inscribed polygon $A_I = \frac{nr^2}{2} \times \sin \frac{360°}{n}$

Area of circumscribed polygon $A_C = nr^2 \times \tan \frac{180°}{n}$




**Use a calculator to complete the chart below for a unit circle (a circle of radius 1).**

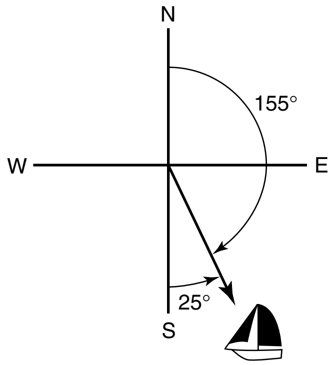
| | Number of Sides | Area of Inscribed Polygon | Area of Circle minus Area of Polygon | Area of Circumscribed Polygon | Area of Polygon minus Area of Circle |
|---|---|---|---|---|---|
| | 3 | 1.2990381 | 1.8425545 | 5.1961524 | 2.054597 |
| **1.** | 4 | | | | |
| **2.** | 8 | | | | |
| **3.** | 12 | | | | |
| **4.** | 20 | | | | |
| **5.** | 24 | | | | |
| **6.** | 28 | | | | |
| **7.** | 32 | | | | |
| **8.** | 1000 | | | | |

**9.** What number do the areas of the circumscribed and inscribed polygons seem to be approaching?

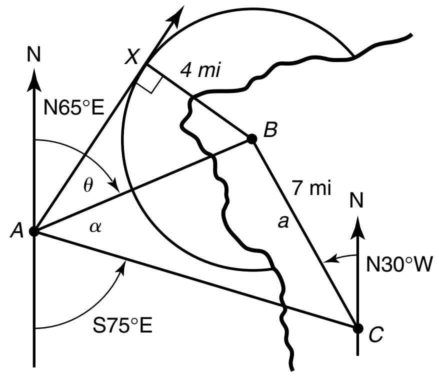
# 13-4 Enrichment

## *Navigation*

The bearing of a boat is an angle showing the direction the boat is heading. Often, the angle is measured from north, but it can be measured from any of the four compass directions. At the right, the bearing of the boat is 155°. Or, it can be described as 25° east of south (S25°E).




*Example* **A boat $A$ sights the lighthouse $B$ in the direction N65°E and the spire of a church $C$ in the direction S75°E. According to the map, $B$ is 7 miles from $C$ in the direction N30°W. In order for $A$ to avoid running aground, find the bearing it should keep to pass $B$ at 4 miles distance.**




In $\triangle ABC$, $\angle\alpha = 180° - 65° - 75°$ or $40°$

$$\angle C = 180° - 30° - (180° - 75°)$$
$$= 45°$$
$$a = 7 \text{ miles}$$

With the Law of Sines,

$$AB = \frac{a \sin C}{\sin \alpha} = \frac{7(\sin 45°)}{\sin 40°} = 7.7 \text{ mi.}$$

The ray for the correct bearing for $A$ must be tangent at $X$ to circle $B$ with radius $BX = 4$. Thus $\triangle ABX$ is a right triangle.

Then $\sin\theta = \frac{BX}{AB} = \frac{4}{7.7} \approx 0.519$. Therefore, $\angle\theta = 31°18'$.

The bearing of $A$ should be $65° - 31°18'$ or $33°42'$.

**Solve the following.**

**1.** Suppose the lighthouse $B$ in the example is sighted at S30°W by a ship $P$ due north of the church $C$. Find the bearing $P$ should keep to pass $B$ at 4 miles distance.

**2.** In the fog, the lighthouse keeper determines by radar that a boat 18 miles away is heading to the shore. The direction of the boat from the lighthouse is S80°E. What bearing should the lighthouse keeper radio the boat to take to come ashore 4 miles south of the lighthouse?

**3.** To avoid a rocky area along a shoreline, a ship at $M$ travels 7 km to $R$, bearing $22°15'$, then 8 km to $P$, bearing $68°30'$, then 6 km to $Q$, bearing $109°15'$. Find the distance from $M$ to $Q$.

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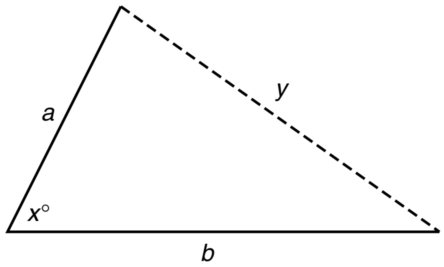
# 13-5 Enrichment

## *The Law of Cosines and the Pythagorean Theorem*

The law of cosines bears strong similarities to the Pythagorean theorem. According to the law of cosines, if two sides of a triangle have lengths $a$ and $b$ and if the angle between them has a measure of $x°$, then the length, $y$, of the third side of the triangle can be found by using the equation

$y^2 = a^2 + b^2 - 2ab \cos x°$.




**Answer the following questions to clarify the relationship between the law of cosines and the Pythagorean theorem.**

**1.** If the value of $x°$ becomes less and less, what number is $\cos x°$ close to?

**2.** If the value of $x°$ is very close to zero but then increases, what happens to $\cos x°$ as $x°$ approaches 90°?

**3.** If $x°$ equals 90°, what is the value of $\cos x°$? What does the equation of $y^2 = a^2 + b^2 - 2ab \cos x°$ simplify to if $x°$ equals 90°?

**4.** What happens to the value of $\cos x°$ as $x°$ increases beyond 90° and approaches 180°?

**5.** Consider some particular value of $a$ and $b$, say 7 for $a$ and 19 for $b$. Use a graphing calculator to graph the equation you get by solving $y^2 = 7^2 + 19^2 - 2(7)(19) \cos x°$ for $y$.

**a.** In view of the geometry of the situation, what range of values should you use for $X$?

**b.** Display the graph and use the TRACE function. What do the maximum and minimum values appear to be for the function?

**c.** How do the answers for part **b** relate to the lengths 7 and 19? Are the maximum and minimum values from part **b** ever actually attained in the geometric situation?

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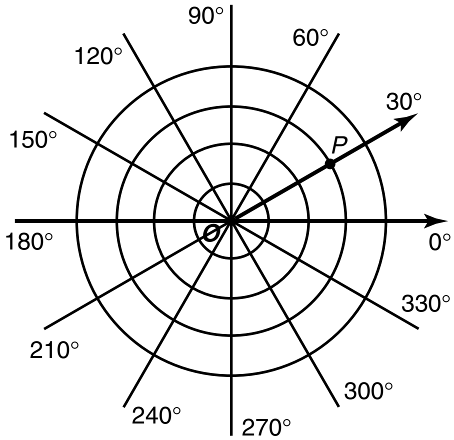
# 13-6 Enrichment

## *Polar Coordinates*

Consider an angle in standard position with its vertex at a point $O$ called the *pole.* Its initial side is on a coordinated axis called the *polar axis.* A point $P$ on the terminal side of the angle is named by the *polar coordinates* $(r, \theta)$ where $r$ is the directed distance of the point from $O$ and $\theta$ is the measure of the angle.




Graphs in this system may be drawn on polar coordinate paper such as the kind shown at the right.

The polar coordinates of a point are not unique. For example, (3, 30°) names point $P$ as well as (3, 390°). Another name for $P$ is (−3, 210°). Can you see why? The coordinates of the pole are $(0, \theta)$ where $\theta$ may be any angle.

**Example** **Draw the graph of the function $r = \cos \theta$. Make a table of convenient values for $\theta$ and $r$. Then plot the points.**

| $\theta$ | 0° | 30° | 60° | 90° | 120° | 150° | 180° |
|---|---|---|---|---|---|---|---|
| $r$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | −1 |

Since the period of the cosine function is 180°, values of $r$ for $\theta > 180°$ are repeated.

**Graph each function by making a table of values and plotting the values on polar coordinate paper.**

**1.** $r = 4$

**2.** $r = 3 \sin \theta$

**3.** $r = 3 \cos 2\theta$

**4.** $r = 2(1 + \cos \theta)$

# 13-7 Enrichment

## *Snell's Law*

Snell's Law describes what happens to a ray of light that passes from air into water or some other substance. In the figure, the ray starts at the left and makes an angle of incidence $\theta$ with the surface.

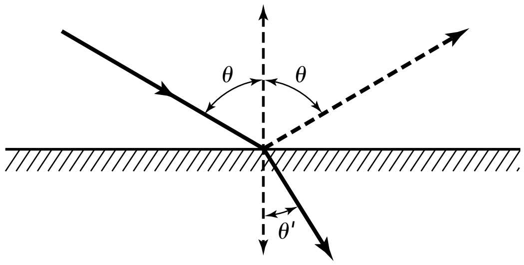
Part of the ray is reflected, creating an angle of reflection $\theta$. The rest of the ray is bent, or refracted, as it passes through the other medium. This creates angle $\theta'$.

The angle of incidence equals the angle of reflection.

The angles of incidence and refraction are related by Snell's Law:

$\sin \theta = k \sin \theta'$

The constant $k$ is called the index of refraction.




| *k* | Substance |
|---|---|
| 1.33 | Water |
| 1.36 | Ethyl alcohol |
| 1.54 | Rock salt and Quartz |
| 1.46–1.96 | Glass |
| 2.42 | Diamond |

**Use Snell's Law to solve the following. Round angle measures to the nearest tenth of a degree.**

**1.** If the angle of incidence at which a ray of light strikes the surface of a window is 45° and $k = 1.6$, what is the measure of the angle of refraction?

**2.** If the angle of incidence of a ray of light that strikes the surface of water is 50°, what is the angle of refraction?

**3.** If the angle of refraction of a ray of light striking a quartz crystal is 24°, what is the angle of incidence?

**4.** The angles of incidence and refraction for rays of light were measured five times for a certain substance. The measurements (one of which was in error) are shown in the table. Was the substance glass, quartz, or diamond?

| $\theta$ | 15° | 30° | 40° | 60° | 80° |
|---|---|---|---|---|---|
| $\theta'$ | 9.7° | 16.1° | 21.2° | 28.6° | 33.2° |

**5.** If the angle of incidence at which a ray of light strikes the surface of ethyl alcohol is 60°, what is the angle of refraction?

# 14-1 Enrichment

## *Blueprints*

Interpreting blueprints requires the ability to select and use trigonometric functions and geometric properties. The figure below represents a plan for an improvement to a roof. The metal fitting shown makes a 30° angle with the horizontal. The vertices of the geometric shapes are *not* labeled in these plans. Relevant information must be selected and the appropriate function used to find the unknown measures.

**Example** **Find the unknown measures in the figure at the right.**

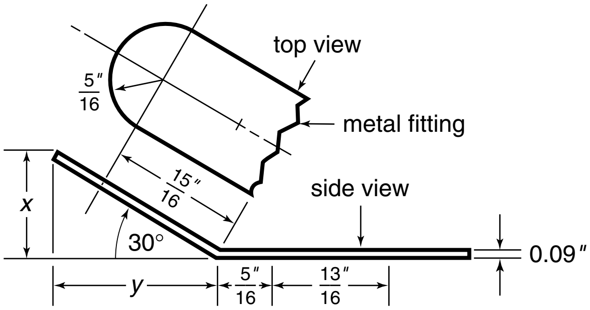
The measures $x$ and $y$ are the legs of a right triangle.

The measure of the hypotenuse is $\frac{15}{16}$ in. + $\frac{5}{16}$ in. or $\frac{20}{16}$ in.

$\frac{y}{\frac{20}{16}} = \cos 30°$ $\qquad$ $\frac{x}{\frac{20}{16}} = \sin 30°$

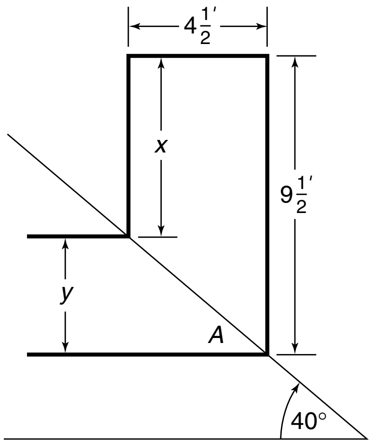
$y = 1.08$ in. $\qquad$ $x = 0.63$ in.

**Roofing Improvement**

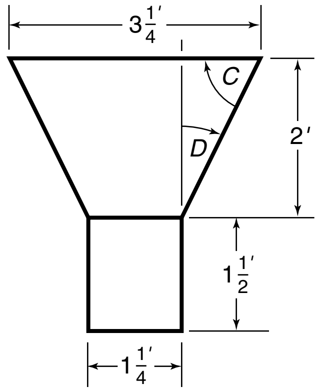



**Find the unknown measures of each of the following.**

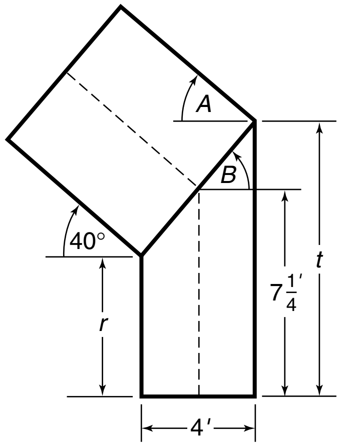
**1.** Chimney on roof




**2.** Air vent




**3.** Elbow joint



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# 14-2 Enrichment

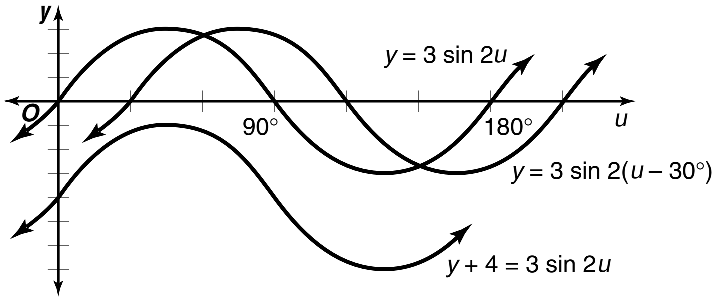
## *Translating Graphs of Trigonometric Functions*

Three graphs are shown at the right:

$y = 3 \sin 2\theta$

$y = 3 \sin 2(\theta - 30°)$

$y + 4 = 3 \sin 2\theta$



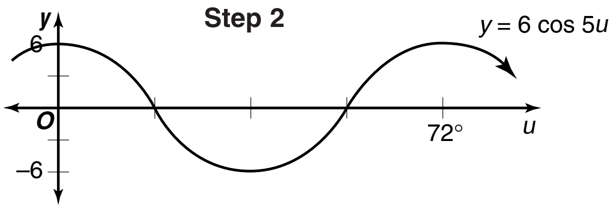

Replacing $\theta$ with $(\theta - 30°)$ translates the graph to the right. Replacing $y$ with $y + 4$ translates the graph 4 units down.

**Example** **Graph one cycle of $y = 6 \cos (5\theta + 80°) + 2$.**

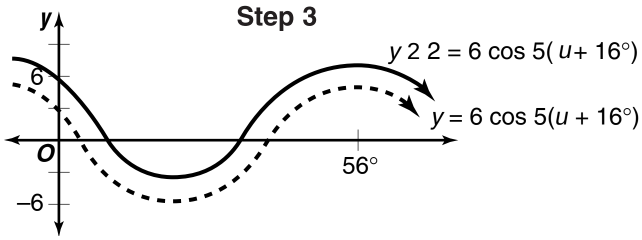
**Step 1** Transform the equation into the form $y - k = a \cos b(\theta - h)$.

$y - 2 = 6 \cos 5(\theta + 16°)$

**Step 2** Sketch $y = 6 \cos 5\theta$.




**Step 3** Translate $y = 6 \cos 5\theta$ to obtain the desired graph.




**Sketch these graphs on the same coordinate system.**

**1.** $y = 3 \sin 2(\theta + 45°)$ **2.** $y - 1 = 3 \sin 2\theta$ **3.** $y + 5 = 3 \sin 2(\theta + 90°)$

**On another piece of paper, graph one cycle of each curve.**

**4.** $y = 2 \sin 4(\theta - 50°)$

**5.** $y = 5 \sin (3\theta + 90°)$

**6.** $y = 6 \cos (4\theta + 360°) + 3$

**7.** $y = 6 \cos 4\theta + 3$

**8.** The graphs for problems 6 and 7 should be the same. Use the sum formula for cosine of a sum to show that the equations are equivalent.

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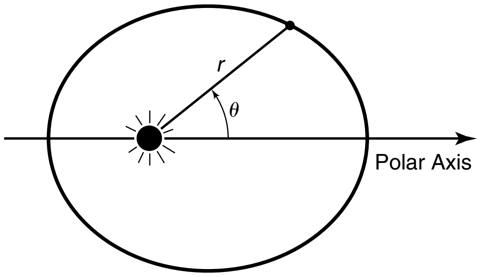
# 14-3 Enrichment

## *Planetary Orbits*

The orbit of a planet around the sun is an ellipse with the sun at one focus. Let the pole of a polar coordinate system be that focus and the polar axis be toward the other focus. The polar equation of an ellipse is




$r = \frac{2ep}{1 - e\cos\theta}$. Since $2p = \frac{b^2}{c}$ and $b^2 = a^2 - c^2$,

$2p = \frac{a^2 - c^2}{c} = \frac{a^2}{c}\left(1 - \frac{c^2}{a^2}\right)$. Because $e = \frac{c}{a}$,

$2p = a\left(\frac{a}{c}\right)\left(1 - \left(\frac{c}{a}\right)^2\right) = a\left(\frac{1}{e}\right)(1 - e^2)$.

Therefore $2ep = a(1 - e^2)$. Substituting into the polar equation of an ellipse yields an equation that is useful for finding distances from the planet to the sun.

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

Note that $e$ is the eccentricity of the orbit and $a$ is the length of the semi-major axis of the ellipse. Also, $a$ is the mean distance of the planet from the sun.

**Example** **The mean distance of Venus from the sun is $67.24 \times 10^6$ miles and the eccentricity of its orbit is .006788. Find the minimum and maximum distances of Venus from the sun.**

The minimum distance occurs when $\theta = \pi$.

$$r = \frac{67.24 \times 10^6(1 - 0.006788^2)}{1 - 0.006788\cos\pi} = 66.78 \times 10^6 \text{ miles}$$

The maximum distance occurs when $\theta = 0$.

$$r = \frac{67.24 \times 10^6(1 - 0.006788^2)}{1 - 0.006788\cos 0} = 67.70 \times 10^6 \text{ miles}$$

**Complete each of the following.**

1. The mean distance of Mars from the sun is $141.64 \times 10^6$ miles and the eccentricity of its orbit is 0.093382. Find the minimum and maximum distances of Mars from the sun.

2. The minimum distance of Earth from the sun is $91.445 \times 10^6$ miles and the eccentricity of its orbit is 0.016734. Find the mean and maximum distances of Earth from the sun.

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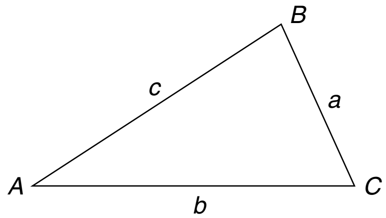
# 14-4 Enrichment

## *Heron's Formula*

Heron's formula can be used to find the area of a triangle if you know the lengths of the three sides. Consider any triangle $ABC$. Let $K$ represent the area of $\triangle ABC$. Then




$$K = \frac{1}{2}bc \sin A$$

$$K^2 = \frac{b^2c^2 \sin^2 A}{4} \quad \text{Square both sides.}$$

$$= \frac{b^2c^2(1 - \cos^2 A)}{4}$$

$$= \frac{b^2c^2(1 + \cos A)(1 - \cos A)}{4}$$

$$= \frac{b^2c^2}{4}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right) \quad \text{Use the law of cosines.}$$

$$= \frac{b + c + a}{2} \cdot \frac{b + c - a}{2} \cdot \frac{a + b - c}{2} \cdot \frac{a - b + c}{2} \quad \text{Simplify.}$$

Let $s = \frac{a + b + c}{2}$. Then $s - a = \frac{b + c - a}{2}$, $s - b = \frac{a + c - b}{2}$, $s - c = \frac{a + b - c}{2}$.

$$K^2 = s(s - a)(s - b)(s - c) \quad \text{Substitute.}$$

$$K = \sqrt{s(s - a)(s - b)(s - c)}$$

| | |
|---|---|
| **Heron's Formula** | The area of $\triangle ABC$ is $\sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{a + b + c}{2}$. |

**Use Heron's formula to find the area of $\triangle ABC$.**

**1.** $a = 3$, $b = 4.4$, $c = 7$

**2.** $a = 8.2$, $b = 10.3$, $c = 9.5$

**3.** $a = 31.3$, $b = 92.0$, $c = 67.9$

**4.** $a = 0.54$, $b = 1.32$, $c = 0.78$

**5.** $a = 321$, $b = 178$, $c = 298$

**6.** $a = 0.05$, $b = 0.08$, $c = 0.04$

**7.** $a = 21.5$, $b = 33.0$, $c = 41.7$

**8.** $a = 2.08$, $b = 9.13$, $c = 8.99$

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# 14-5 Enrichment

## *Identities for the Products of Sines and Cosines*

By adding the identities for the sines of the sum and difference of the measures of two angles, a new identity is obtained.

$$\begin{array}{ll} & \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ & \underline{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \\ (i) & \sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta \end{array}$$

This new identity is useful for expressing certain products as sums.

**Example** **Write $\sin 3\theta \cos\theta$ as a sum.**

In the identity let $\alpha = 3\theta$ and $\beta = \theta$ so that

$2\sin 3\theta\cos\theta = \sin(3\theta + \theta) + \sin(3\theta - \theta)$. Thus,

$\sin 3\theta\cos\theta = \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta$.

By subtracting the identities for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$, a similar identity for expressing a product as a difference is obtained.

$(ii)$ $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$

**Solve.**

**1.** Use the identities for $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$ to find identities for expressing the products $2\cos\alpha\cos\beta$ and $2\sin\alpha\sin\beta$ as a sum or difference.

**2.** Find the value of $\sin 105° \cos 75°$ without using tables.

**3.** Express $\cos\theta\sin\frac{\theta}{2}$ as a difference.

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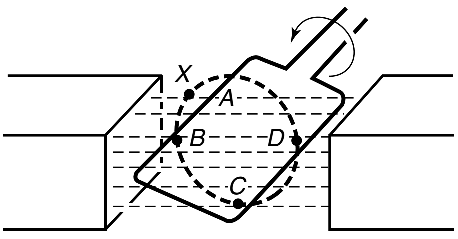
# 14-6 Enrichment

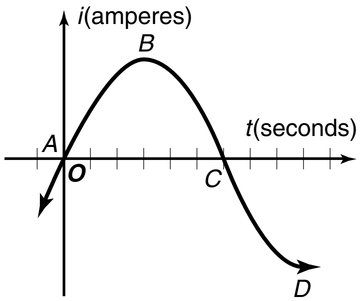
## *Alternating Current*

The figure at the right represents an alternating current generator. A rectangular coil of wire is suspended between the poles of a magnet. As the coil of wire is rotated, it passes through the magnetic field and generates current.




As point $X$ on the coil passes through the points $A$ and $C$, its motion is along the direction of the magnetic field between the poles. Therefore, no current is generated. However, through points $B$ and $D$, the motion of $X$ is perpendicular to the magnetic field. This induces maximum current in the coil. Between $A$ and $B$, $B$ and $C$, $C$ and $D$, and $D$ and $A$, the current in the coil will have an intermediate value. Thus, the graph of the current of an alternating current generator is closely related to the sine curve.

The maximum current may have a positive or negative value.




The actual current, $i$, in a household current is given by $i = I_M \sin(120\pi t + \alpha)$ where $I_M$ is the maximum value of the current, $t$ is the elapsed time in seconds, and $\alpha$ is the angle determined by the position of the coil at time $t_n$.

**Example** **If $\alpha = \frac{\pi}{2}$, find a value of $t$ for which $i = 0$.**

If $i = 0$, then $I_M \sin(120\pi t + \alpha) = 0$. $\quad i = I_M \sin(120\pi t + \alpha)$

Since $I_M \neq 0$, $\sin(120\pi t + \alpha) = 0$. $\quad$ If $ab = 0$ and $a \neq 0$, then $b = 0$.

Let $120\pi t + \alpha = s$. Thus, $\sin s = 0$.

$s = \pi$ because $\sin \pi = 0$.

$120\pi t + \alpha = \pi$ $\quad$ Substitute $120\pi t + \alpha$ for $s$.

$120\pi t + \frac{\pi}{2} = \pi$ $\quad$ Substitute $\frac{\pi}{2}$ for $\alpha$.

$= \frac{1}{240}$ $\quad$ Solve for $t$.

This solution is the first positive value of $t$ that satisfies the problem.

**Using the equation for the actual current in a household circuit, $i = I_M \sin(120\pi t + \alpha)$, solve each problem. For each problem, find the first positive value of $t$.**

**1.** If $\alpha = 0$, find a value of $t$ for which $i = 0$.

**2.** If $\alpha = 0$, find a value of $t$ for which $i = +I_M$.

**3.** If $\alpha = \frac{\pi}{2}$, find a value of $t$ for which $i = -I_M$.

**4.** If $\alpha = \frac{\pi}{4}$, find a value of $t$ for which $i = 0$.

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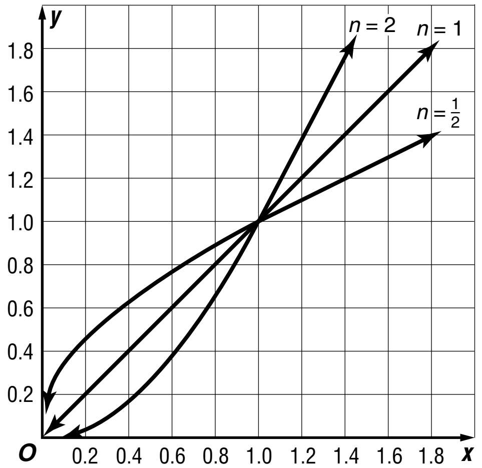
# 14-7 Enrichment

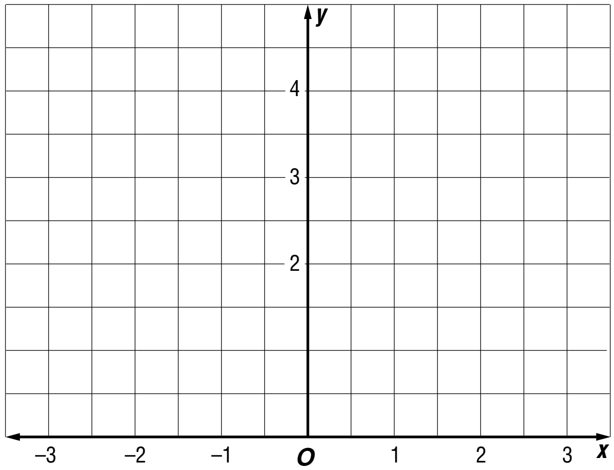
## *Families of Curves*

**Use these graphs for the problems below.**

The Family $y \quad x^n$




The Family $y \quad e^{mx}$




1. Use the graph on the left to describe the relationship among the curves $y = x^{\frac{1}{2}}$, $y = x^1$, and $y = x^2$.

2. Graph $y = x^n$ for $n = \frac{1}{10}, \frac{1}{4}$, 4, and 10 on the grid with $y = x^{\frac{1}{2}}$, $y = x^1$, and $y = x^2$.

3. Which two regions in the first quadrant contain no points of the graphs of the family for $y = x^n$?

4. On the right grid, graph the members of the family $y = e^{mx}$ for which $m = 1$ and $m = -1$.

5. Describe the relationship among these two curves and the $y$-axis.

6. Graph $y = e^{mx}$ for $m = 0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm2$, and $\pm4$.